Lecture 11: Overview

- Maximum Flow
Problem

- Have a directed graph
- Use to model flow of material from source to sink
- Edges contain maximum quantity of material that may flow across
- Use $c(u,v)$ to store capacity of edge from $u$ to $v$
Constraints

• Capacity constraint:
  • For all $u,v \in V$, $f(u,v) \leq c(u,v)$

• Skew Symmetry:
  • For all $u,v \in V$, $f(u,v) = -f(v,u)$

• Flow Conservation:
  • For all $u \in V - \{s,t\}$, we require

$$\sum_{v \in V} f(u,v) = 0$$
Optimize

- Maximize the flow through the network

\[ |f| = \sum_{v \in V} f(s,v) \]
Multiple Sources/Sinks

• Can convert to single source/sink
• Idea:
  • Add supersource s and supersink t
  • Connect these to sources and sinks via edges with an infinite capacity
Ford-Fulkerson Method

FORD-FULKERSON-METHOD(G, s, t)
1 initialize flow f to 0
2 while there exist an augmenting path p
3 do augment flow f along p
4 return f
Residual Capacities

- Amount of additional flow we can push from u to v before exceeding the capacity $c(u, v)$
- $c_f(u, v) = c(u, v) - f(u, v)$
- Augmenting path is a path through the residual network
Max flow Min-cut Theorem

• If f is a flow in a flow network $G=(V, E)$ with source s and sink t, then the following conditions are equivalent:
  1. f is a maximum flow in G
  2. The residual network $G_f$ contains no augmenting paths
  3. $|f|=c(S, T)$ for some cut (S,T) of G
What is the complexity of Ford-Fulkerson method?

- If the values are irrational and choices made poorly, it may not terminate!!!
- Can even take a long time with integers:
Edmonds-Karp Algorithm

- Suppose we always choose the shortest path from s to t in the residual network?
- Complexity is $O(VE^2)$
- Why:
  - Each augmentation has complexity $O(E)$
  - Numbers of augmentations is $O(VE)$
    - Each edge can be a critical edge at most $|V|/2-1$ times
    - Once edge is critical, its flow is saturated
    - Must appear in path the other way next
    - Since previous path is shortest, this reversing path must be two edges longer
\[ d_{f'}(s,v) = d_f(s,u) + 1 \]
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\[ \geq d_f(s,v) + 1 \text{ (paths increase monotonically)} \]
\[ = d_f(s,u) + 2 \]
Maximum Bipartite Matching

- Idea: Have bipartite graph
- Want to pick maximum matching
- Matching is a subset of edges $M \subseteq E$ such that for all vertices $v \in V$ that at most of edge of $M$ is incident on $V$
- How is this related to maximal flow problem?
Idea:

• Add source and sink
• Connect source to all nodes on “left” side with edges of capacity 1
• Connect all nodes on “right” side to sink with edges of capacity 1
• Take all edges in original bipartite graph to have capacity 1