Lecture 11: Overview

• Maximum Flow
Problem

- Have a directed graph
- Use to model flow of material from source to sink
- Edges contain maximum quantify of material that may flow across
- Use $c(u,v)$ to store capacity of edge from $u$ to $v$
Constraints

• Capacity constraint:
  • For all $u,v \in V$, $f(u,v) \leq c(u,v)$

• Skew Symmetry:
  • For all $u,v \in V$, $f(u,v) = -f(v, u)$

• Flow Conservation:
  • For all $u \in V - \{s,t\}$, we require
Optimize

- Maximize the flow through the network
Multiple Sources/Sinks

- Can convert to single source/sink
- Idea:
  - Add supersource $s$ and supersink $t$
  - Connect these to sources and sinks via edges with an infinite capacity
Ford-Fulkerson Method

FORD-FULKERSON-METHOD(G, s, t)
1 initialize flow f to 0
2 while there exist an augmenting path p
3 do augment flow f along p
4 return f
Residual Capacities

- Amount of additional flow we can push from $u$ to $v$ before exceeding the capacity $c(u, v)$
- $c_f(u, v) = c(u, v) - f(u, v)$
- Augmenting path is a path through the residual network
Max flow Min-cut Theorem

• If $f$ is a flow in a flow network $G=(V, E)$ with source $s$ and sink $t$, then the following conditions are equivalent:

1. $f$ is a maximum flow in $G$
2. The residual network $G_f$ contains no augmenting paths
3. $|f| = c(S, T)$ for some cut $(S, T)$ of $G$
What is the complexity of Ford-Fulkerson method?

- If the values are irrational and choices made poorly, it may not terminate!!!
- Can even take a long time with integers:
Edmonds-Karp Algorithm

• Suppose we always choose the shortest path from s to t in the residual network?
• Complexity is $O(VE^2)$
• Why:
  • Each augmentation has complexity $O(E)$
  • Numbers of augmentations is $O(VE)$
    • Each edge can be a critical edge at most $|V|/2-1$ times
    • Once edge is critical, its flow is saturated
    • Must appear in path the other way next
    • Since previous path is shortest, this reversing path must be two edges longer
Cont

\[ d'_f(s,v) = d_f(s,u) + 1 \]
\[ d'_f(s,u) = d'_f(s,v) + 1 \]

\[ \geq d_f(s,v) + 1 \text{ (paths increase monotonically)} \]
\[ = d_f(s,u) + 2 \]
Maximum Bipartite Matching

- Idea: Have bipartite graph
- Want to pick maximum matching
- Matching is a subset of edges $M \subseteq E$ such that for all vertices $v \in V$ that at most of edge of $M$ is incident on $V$
- How is this related to maximal flow problem?
Idea:

- Add source and sink
- Connect source to all nodes on “left” side with edges of capacity 1
- Connect all nodes on “right” side to sink with edges of capacity 1
- Take all edges in original bipartite graph to have capacity 1