1. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space. If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$.

**Proof.** Since $\bigcup_{i=1}^{n} \overline{A_i}$ is a finite union of closed sets, $\bigcup_{i=1}^{n} \overline{A_i}$ is closed and it contains $B_n$. We know that $\overline{B_n}$ is the intersection of all closed sets containing $B_n$, so we have $\overline{B_n} \subseteq \bigcup_{i=1}^{n} \overline{A_i}$. Conversely, suppose $a \in \overline{A_i}$ for some $i$. Let $\varepsilon > 0$ be given. Then $B(a; \varepsilon) \cap A_i \neq \emptyset$, and hence $B(a; \varepsilon) \cap \bigcup_{i=1}^{n} A_i = B(a; \varepsilon) \cap B_n \neq \emptyset$ which implies $a$ adheres to $B_n$. Hence, we have containment in the other direction which completes the proof. ■

Remark: Consider the following argument. Let $x \in \overline{B_n}$ and let $\varepsilon > 0$ be given. Then $B(x; \varepsilon) \cap B_n \neq \emptyset$. Hence, $B(x; \varepsilon) \cap \bigcup_{i=1}^{n} A_i$ is nonempty, which implies $B(x; \varepsilon) \cap A_k \neq \emptyset$ for some $k$. Therefore, $x$ adheres to $A_k$.

There is a problem with this argument that went overlooked in many of the solutions on the exam. We cannot conclude from here that $x$ adheres to $A_k$. In fact, $k$ may vary with $\varepsilon$, and as $\varepsilon$ shrinks we may have to change to another $k$. Consider, for example, the set $S = (0, 1) \cup (1, 2)$. If $\varepsilon = 2$ and $x = 0$ then $B(x; \varepsilon) \cap (1, 2) \neq \emptyset$ but $x$ does not adhere to $(1, 2)$. Notice that this is precisely the reason that, in general, the union of an infinite number of closed sets is not closed, and this is where the assumption that we have a finite union becomes important. What we have actually shown is that for each $\varepsilon > 0$ there exists a $k$ such that $B(x; \varepsilon) \cap A_k \neq \emptyset$, but from this statement we cannot conclude that for each $\varepsilon > 0$ $B(x; \varepsilon) \cap A_{k_0} \neq \emptyset$ for a particular $k_0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sin \frac{x^2}{2} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$  

(a) Is the function $f$ continuous at 0?

(b) Is it uniformly continuous on $S_1 = (0, \infty)$?

(c) Is it uniformly continuous on $S_2 = (-1, \infty)$?

Prove all your statements. You can use the trig identity: $\sin \alpha - \sin \beta = 2 \cos \left(\frac{\alpha + \beta}{2}\right) \sin \left(\frac{\alpha - \beta}{2}\right)$.

**Solution.**

(a) We claim $f(x)$ is not continuous at 0. Indeed, consider the sequence $x_n = 1/\left(4(n+1)^2 \pi^2\right)$. Then $\lim_{n \to \infty} x_n \to 0$ but $f(x_n) = 1$ for each $n$ which implies $\lim_{n \to \infty} f(x_n) = 1 \neq 0 = f(0) = f(\lim_{n \to \infty} x_n)$.

(b) We will prove that $f$ is not uniformly continuous on $S_1$. To see this, take $\varepsilon = 1$ and let $\delta > 0$ be given. The sequence $x_n = \frac{1}{2(n+1)^2 \pi}$ is convergent and in particular is Cauchy, so choose $N$ large enough that $|x_N - x_{N+1}| < \delta$. Then

$$|f(x_N) - f(x_{N+1})| = 2 \geq \varepsilon.$$

(c) We claim $f(x)$ is uniformly continuous on $(.1, \infty)$. Let $\varepsilon > 0$ be given. We know that $\lim_{x \to 0} \sin x = 0$, so choose $\delta'$ so small that $|\sin x| < \varepsilon/2$ whenever $|x| < \delta'$, and let $\delta = \frac{1}{100} \delta'$. Let $x, y \in (.1, \infty)$ so that $|x - y| < \delta$. Then

$$\left|\frac{\frac{1}{x} - \frac{1}{y}}{2}\right| \leq \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| \leq \left|\frac{\delta}{|xy|}\right| \leq 100\delta = \delta'.$$

Hence,

$$\left|\sin \frac{1}{x} - \sin \frac{1}{y}\right| = 2 \left|\cos \left(\frac{\frac{1}{x} + \frac{1}{y}}{2}\right)\right| \left|\sin \left(\frac{\frac{1}{x} - \frac{1}{y}}{2}\right)\right| \leq 2 \left|\sin \left(\frac{\frac{1}{x} - \frac{1}{y}}{2}\right)\right| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$
3. Consider the set in \(\mathbb{R}^2\), \(S = A \cup B \cup C\), where \(A = \{(x, y) : x \in (0, \infty), y = x/(x+1)\}\), \(B = \{(x, 1) : x \in (0, \infty)\}\), \(C = \{(x, y) : x \in (0, \infty), y = 1 + 1/x\}\).

(a) Is \(S\) open?
(b) Is \(S\) compact?
(c) Is \(S\) connected?

Prove all your statements.

Solution.

(a) We claim \(S\) is not open (in \(\mathbb{R}^2\)). Indeed, \((1, 1) \in B \subseteq S\), but for any \(0 < \varepsilon < 1\) the ball \(B((1, 1), \varepsilon)\) contains the point \((1, 1 + \varepsilon/2) \notin S\). If \(\varepsilon \geq 1\), the point \((1, 5/4) \in B((1, 1), \varepsilon)\) but \((1, 5/4) \notin S\).

(b) \(S\) is not bounded so in particular \(S\) cannot be compact.

(c) \(S\) is not connected. We already have \(S = A \cup B \cup C\) where \(A, B\) and \(C\) are nonempty, so if we can show \(B\) and \(A \cup C\) are disjoint and open in \(S\) we will be done. Since \(x/(x+1) \neq 1\) and \(1 + 1/x \neq 1\) for each \(x \in (0, \infty)\), we have the disjointness. Let \((a, 1) \in B\), and let \(\varepsilon > 0\) be small enough so that \(\varepsilon < 1, x/(x+1) < 1 - \varepsilon\) for each \(x \leq a + 1\) and \(1 + 1/x > 1 + \varepsilon\) for each \(x \leq a + 1\) (we can find such an \(\varepsilon\) here since \(x/(x+1)\) is an increasing function always less than 1 and \(1 + 1/x\) is a decreasing function always greater than 1). In this case, \(B_S((a, 1); \varepsilon) \subseteq B\). One can show that \(A \cup C\) is open in a very similar way, which is left to the reader.