Math 205A

Partial solutions for homework assignment 5
Please email Josh Whitney at jwhitney {at} uci {dot} edu with any errors or oversights.

Exercise 1 (3.27) Consider the following two metrics in $\mathbb{R}^n$:

$$d_1(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|, \quad d_2(x,y) = \sum_{i=1}^{n} |x_i - y_i|.$$  

In each of the following metric spaces prove that the ball $B(a;r)$ has the geometric appearance indicated:

1. In $(\mathbb{R}^2,d_1)$, a square with sides parallel to the coordinate axes.
2. In $(\mathbb{R}^2,d_2)$, a square with diagonals parallel to the axes.

Solution. Throughout this solution we will assume $a$ is the origin to simplify the notation.

1. For notational simplicity, we assume here that $a$ is the origin. We claim that $B(a;r) = S$, where $S = \{(x,y) \in \mathbb{R}^2 : |x| < r, |y| < r\}$, which is a square centered at the origin with side length $2r$. Certainly if $(x,y) \in S$ then $d((x,y),(0,0)) = \max(|x|, |y|) < r$ so $(x,y) \in B(a;r)$. Conversely, if $(x,y) \in B(a,r)$ then $|x|, |y| \leq \max(|x|, |y|) < r$, so we have $(x,y) \in S$.

2. By definition, $B(a;r) = \{(x,y) \in \mathbb{R}^2 : |x| + |y| < r\}$. Therefore, $(x,y)$ is on the boundary of the ball is when $|x| + |y| = r$. The boundary is then composed of four line segments

$$y = r - x, \quad 0 \leq x \leq r$$
$$y = -r + x, \quad 0 \leq x \leq r$$
$$y = r + x, \quad -r \leq x \leq 0$$
$$y = -r - x, \quad -r \leq x \leq 0.$$

This is precisely the square of side length $2r$ whose diagonals lie along the coordinate axes.

Exercise 2 (3.32) Prove that every Euclidean space $\mathbb{R}^k$ is separable.

Proof. We claim that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. Let $x = (x_1, ..., x_k) \in \mathbb{R}^k$ and let $\varepsilon > 0$ be given. We know that the rationals are dense in the reals, so for each $x_i$ choose a rational number $q_i$ such that $|x_i - q_i| < \varepsilon \sqrt{1/k}$. If $q = (q_1, ..., q_k)$ then

$$||x - q||^2 = \sum_{i=1}^{k} (x_i - q_i)^2 = \sum_{i=1}^{k} \varepsilon^2/k = \varepsilon^2.$$

Therefore, $||x - q|| < \varepsilon$. 

Exercise 3 (3.34) Prove that the Lindelöf covering theorem is valid in any separable metric space.

Proof. This proof follows very closely the proof of theorem 3.28. Let $M$ be a separable metric space, $A \subseteq M$ and let $\mathcal{F}$ be an open covering of $A$. Suppose $B = \{b_1, b_2, ...\}$ is a countable subset of $M$ such that $B$ is dense in $M$, and let $\mathcal{G}$ be the countable collection of all balls in $M$ with centers at some $b_i$ and rational radii. Assume $x \in M$. Then there is an open set $S$ in $\mathcal{F}$ such that $x \in S$. Take $\varepsilon$ so small that $B(x;\varepsilon) \subseteq S$. Choose a rational $r(x) > 0$ such that $r(x) < \varepsilon/2$. Since $B$ is dense in $M$, there is some $b_i(x) \in B$ for which $d(x, b_i(x)) < r/2$. Now, $B(b_i(x), r(x)/2)$ is an element of $\mathcal{G}$, and further $B(b_i(x), r(x)/2) \subseteq S$. Then the set of all $B(b_i(x), r(x)/2)$ as $x$ ranges over the elements of $A$ is a countable covering of $A$. To get a countable subset of $\mathcal{F}$ which covers $A$, we simply correlate to each set $B(b_i(x), r(x)/2)$ one of the sets $S$ of $\mathcal{F}$ which contains $B(b_i(x), r(x)/2)$. 

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Exercise 4 (3.39) If $S$ is closed and $T$ is compact, then $S \cap T$ is compact.

Proof. Since $T$ is compact, $T$ is also closed in $M$. Therefore, $S \cap T$ is closed in $T$ by theorem 3.34. By theorem 3.39, $S \cap T$ is a compact subset of $T$, which implies that $S \cap T$ is a compact subset of $M$ by exercise 3.38 which was solved in discussion on 11 – 6 – 07. ■

Exercise 5 (3.40) The intersection of an arbitrary collection of compact subsets of $M$ is compact.

Proof. Let $\{A_i\}_{i \in I}$ for some indexing set $I$ be a collection of compact subsets of $M$, and let $A$ be their intersection. Suppose $F$ is an open cover of $A$. Since each $A_i$ is closed in $M$, so too is $A$. Therefore, if we define $A' = M - A$ then $A'$ is open. The idea here is to augment our open cover $F$ with $A'$ so that we have an open cover of each $A_i$. Let $F' = F \cup \{A'\}$. Choose any $i_0 \in I$. Then $F'$ is an open cover of $A_{i_0}$ and hence has a finite subcover $G$. We can assume $G$ includes $A'$, for if not, throwing it in doesn’t change that fact that we have a cover. In particular, $G$ covers $A$, but so too does $G - \{A'\}$ which is a finite subset of $F$. Hence, $A$ is compact. ■

Exercise 6 (3.50) Give an example in which $\text{int} A = \text{int} B = \emptyset$, but $\text{int}(A \cup B) = M$.

Solution. Consider the standard Euclidean space $\mathbb{R}$ and take $A = \mathbb{Q}$ and $B = \mathbb{R} - \mathbb{Q}$. Since any ball in $\mathbb{R}$ contains both rational and irrational points, $\text{int} A = \text{int} B = \emptyset$. However, $A \cup B = \mathbb{R}$, so $\text{int}(A \cup B) = \mathbb{R}$. ■

Exercise 7 (3.52) If $\overline{A} \cap \overline{B} = \emptyset$, then $\partial(A \cup B) = \partial A \cup \partial B$.

Proof. Suppose $x \in \partial A \cup \partial B$. Suppose first that $x \in \partial A$. Then $x$ adheres to $A$ which implies $x$ adheres to $A \cup B$. Notice that $(A \cup B)^c = A^c \cap B^c$. Let $\varepsilon > 0$ be given. We must show that $B(x; \varepsilon) \cap (A^c \cap B^c) \neq \emptyset$. Since $x$ adheres to $A$, $\overline{A} \cap \overline{B} = \emptyset$, $x \notin \overline{B}$. Shrinking $\varepsilon$ if necessary, we can assume that $B(x; \varepsilon) \cap B = \emptyset$. Therefore, $B(x; \varepsilon) \subseteq B^c$. Since $x$ adheres to $A^c$, $B(x; \varepsilon) \cap A^c \neq \emptyset$, say $y \in B(x; \varepsilon) \cap A^c$. Then $y \in B^c$ also, and hence $y \in B(x; \varepsilon) \cap (A^c \cap B^c)$, as desired. A similar argument shows the same result if $x \in \partial B$.

Conversely, suppose that $x \in \partial(A \cup B)$. Then $x$ adheres to $A \cup B$ and $A^c \cap B^c$, the latter of which implies $x$ adheres to both $A^c$ and $B^c$. If $x$ adheres to $A$, then $x \in \partial A$ and we are done. Suppose, on the other hand, that $x$ does not adhere to $A$. We will show that $x$ adheres to $B$. To this end, let $\varepsilon > 0$ be given. By shrinking $\varepsilon$ if necessary, we can assume that $B(x; \varepsilon) \cap A = \emptyset$. But, $x$ adheres to $A \cup B$ so we must have $B(x; \varepsilon) \cap B \neq \emptyset$. Therefore, $x$ adheres to $B$. In any case, $x \in \partial A \cup \partial B$. ■