**Math 205A**

Partial solutions for homework assignment 6
Please email Josh Whitney at jwhitney {at} uci {dot} edu with any errors or oversights.

**Exercise 1 (4.10)** Let $f$ be defined on an open interval $(a, b)$ and assume $x \in (a, b)$. Consider the two statements

a) $\lim_{h \to 0} |f(x + h) - f(x)| = 0$;  

b) $\lim_{h \to 0} |f(x + h) - f(x - h)| = 0$.

Prove that a) always implies b), and give an example in which b) holds but a) does not.

**Solution.** Suppose a) holds and let $\varepsilon > 0$ be given. Then there is some $\delta > 0$ for which

$$ |f(x + h) - f(x)| < \varepsilon / 2 $$

whenever $x + h \in (a, b)$ and $0 < |h| < \delta$. Therefore, if $x + h, x - h \in (a, b)$ and $0 < |h| < \delta$,

$$ |f(x + h) - f(x - h)| = |f(x + h) - f(x)| + |f(x - h) - f(x)| < \varepsilon / 2 + \varepsilon / 2 = \varepsilon. $$

This shows that $\lim_{h \to 0} |f(x + h) - f(x - h)| = 0$, which is b).

Let

$$ f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} $$

Then in this case $(a, b) = \mathbb{R}$ and

$$ \lim_{h \to 0} |f(0 + h) - f(0 - h)| = 0 $$

since $f(0 + h) = 1$ for every $h \neq 0$. On the other hand,

$$ \lim_{h \to 0} |f(0 + h) - f(0)| = 1. $$

**Exercise 2 (4.16)** Let $f$, $g$, and $h$ be defined on $[0, 1]$ as follows:

- $f(x) = g(x) = h(x) = 0$, whenever $x$ is irrational
- $f(x) = 1$ and $g(x) = x$, whenever $x$ is rational
- $h(x) = 1/n$, if $x$ is the rational number $m/n$ (in lowest terms)
- $h(0) = 1$.

Prove that $f$ is not continuous anywhere in $[0, 1]$, that $g$ is continuous only at $x = 0$, and that $h$ is continuous only at the irrational points of $[0, 1]$.

**Proof.** We prove here only the statement about $h$. The statements about $f$ and $g$ can be proved using a similar argument which is a little easier. First, let $m/n$ be a rational number. Then $h(m/n) = 1/n$. Let $\varepsilon = 1/(2n)$. Then for each $\delta > 0$ there is some irrational number $\xi \in B(m/n, \delta) \cap [0, 1]$ by the density of the irrationals in the reals. We have

$$ |h(m/n) - h(\xi)| = 1/n \geq 1/(2n). $$

Therefore, $h$ is not continuous at $m/n$.

Let $x$ be an irrational number and let $\varepsilon > 0$ be given. Let $n \in \mathbb{Z}^+$ be such that $1/n < \varepsilon$. Let $S = \{p/q : 0 \leq p \leq q$ and $1 \leq q \leq n\}$. Since $S$ is a finite set, we can define

$$ \delta = \frac{1}{2} \min_{p/q \in S} ((p/q - x)). $$

Let $y \in [0, 1]$ with $0 < |x - y| < \delta$. If $y \in \mathbb{Q}^c$ then $|f(x) - f(y)| = 0 < \varepsilon$. Otherwise, write $y = p/q$. Then by the way we constructed $\delta$, $y$ cannot have denominator less or equal $n$, for then $y \in S$. Hence, $q \geq n$ and so

$$ |f(x) - f(y)| = \frac{1}{q} \leq \frac{1}{n} < \varepsilon, $$

completing the proof. □
Exercise 3 (4.21) Let \( f : S \to \mathbb{R} \) be continuous on an open set \( S \) in \( \mathbb{R}^n \), assume that \( p \in S \), and assume that \( f(p) > 0 \). Prove that there is an \( n \)-ball \( B(p;r) \) such that \( f(x) > 0 \) for every \( x \) in the ball.

Proof. Consider the set \( V = \{(f(p)/2, 2f(p))\} \subseteq \mathbb{R} \), which is open in \( \mathbb{R} \). Let \( U = f^{-1}(V) \). Then by the continuity of \( f \), \( U \) is open in \( S \) which implies \( U \) is open in \( \mathbb{R}^n \). Now, \( p \in U \) so there is some \( r > 0 \) for which \( B(p;r) \subseteq U \). Therefore, \( f(B(p;r)) \subseteq V \), which is to say that \( f(x) \in (f(p)/2, 2f(p)) \) for each \( x \in B(p;r) \). In particular, \( f(x) > 0 \) for each \( x \in B(p;r) \).

Exercise 4 (4.23) Given a function \( f : \mathbb{R} \to \mathbb{R} \), define two sets \( A \) and \( B \) in \( \mathbb{R}^2 \) as follows:

\[
A = \{(x, y) : y < f(x)\}, \quad B = \{(x, y) : y > f(x)\}.
\]

Prove that \( f \) is continuous on \( \mathbb{R} \) if, and only if, both \( A \) and \( B \) are open subsets of \( \mathbb{R}^2 \).

Proof. Suppose first that \( f \) is continuous on \( \mathbb{R} \). Consider the function \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( g(x, y) = (f(x), y) \). We claim \( g(x) \) is continuous on \( \mathbb{R}^2 \). Let \( \varepsilon > 0 \) be given, and let \( (x_0, y_0) \in \mathbb{R}^2 \). Let \( \delta \) be small enough so that \( \delta < \varepsilon/2 \) and \( |f(x) - f(x_0)| < \varepsilon/2 \) whenever \( x \in \mathbb{R} \) and \( 0 < |x - x_0| \leq \delta \). If \( (x, y) \in \mathbb{R}^2 \) with \( 0 < \|(x, y) - (x_0, y_0)\| < \delta \), then in particular \( |x - x_0| < \delta \) and \( |y - y_0| < \delta < \varepsilon/2 \) so

\[
\|g(x, y) - g(x_0, y_0)\| = \sqrt{(f(x) - f(x_0))^2 + (y - y_0)^2} < \varepsilon/\sqrt{2} < \varepsilon.
\]

Now, define \( h : \mathbb{R}^2 \to \mathbb{R} \) by \( h(x, y) = x - y \). Then \( h \) is continuous. Notice that \( B = (h \circ g)^{-1}((0, \infty)) \) and \( A = (h \circ g)^{-1}((\infty, 0)) \), so \( B \) and \( A \) are both open by the continuity of \( h \circ g \) and the fact that \( (0, \infty) \) and \( (-\infty, 0) \) are open.

The converse was proved in discussion. \( \blacksquare \)