Exercise 1 (4.25) Let $f$ be continuous on a compact interval $[a, b]$. Suppose that $f$ has a local maximum at $x_1$ and a local maximum at $x_2$. Show that there must be a third point between $x_1$ and $x_2$ where $f$ has a local minimum.

Proof. We assume here that the author intended for $x_1$ and $x_2$ to be distinct. So, suppose without loss of generality that $x_1 < x_2$. Then $f$ is continuous on the compact interval $[x_1, x_2]$, and by theorem 4.28 there is some $x_0 \in [x_1, x_2]$ for which $f(x_0) = \inf f([x_1, x_2])$. We claim that we can take $x_0$ to be distinct from $x_1$ or $x_2$. If $x_0 = x_1$, then there is some $\varepsilon > 0$ for which $f(x) \leq f(x_1)$ for each $x \in B(x_1; \varepsilon) \cap [a, b]$ by virtue of the fact that $x_1$ is a local maximum of $f$, and $f(x) \geq f(x_0) = f(x_1)$ for each $x \in [x_1, x_2]$. Therefore, $f(x) = f(x_0) = f(x_1)$ for each $x \in B(x_1; \varepsilon) \cap [x_1, x_2]$. Therefore, we can replace $x_0$ by any $x \in (x_1, x_1 + \varepsilon) \cap (x_1, x_2)$. A similar argument holds if $x_0 = x_2$. We claim that $x_0$ is a local minimum of $f$. Indeed, take $\varepsilon = \frac{1}{2} \min \{x_0 - x_1, x_2 - x_0\}$. Then $B(x_0; \varepsilon) \subseteq (x_1, x_2) \subseteq [a, b]$ and $f(x) \geq f(x_0)$ for each $x \in B(x_0; \varepsilon) \cap [a, b]$.

Exercise 2 (4.26) Let $f$ be a real-valued function, continuous on $[0, 1]$, with the following property: For every real $y$, either there is no $x$ in $[0, 1]$ for which $f(x) = y$ or there is exactly one such $x$. Prove that $f$ is strictly monotonic on $[0, 1]$.

Proof. Suppose first that $f(1) > f(0)$. We will show that $f$ is strictly increasing on $[0, 1]$. Let $a \in (0, 1)$. By assumption $f(a) \neq f(0), f(1)$. We claim that $f(0) < f(a) < f(1)$. Indeed, suppose $f(a) < f(0)$. Let $c = \frac{f(0) + f(a)}{2}$. Then by the intermediate value theorem there is some $x_1 \in (0, a)$ for which $f(x_1) = c$. But, $f(a) < f(0) < f(1)$ also, so there is some $x_2 \in (a, 1)$ such that $f(x_2) = c$. This is a contradiction. A similar argument shows that $f(a) < f(1)$. Therefore, $f(0) < f(a) < f(1)$.

Suppose by way of contradiction that $b \in (a, 1]$ with $f(b) \leq f(a)$. By assumption we must have $f(b) < f(a)$. Then by the intermediate value theorem (4.38) there is some $x_1 \in (a, b)$ for which $f(x_1) = c$ where $c = \frac{f(b) + f(a)}{2}$. However, we also must have $f(0) < c < f(a)$, so there is some $x_2 \in (a, 1)$ for which $f(x_2) = c$ also, a contradiction. Therefore, $f(b) > f(a)$, which shows that $f$ is strictly increasing.

Finally, if $f(1) < f(0)$ apply what we just proved to $-f$ to show that $f$ must be decreasing.

Exercise 3 (4.31) Prove that $f$ is continuous on $S$ if, and only if, $f$ is continuous on every compact subset of $S$.

Proof. The implication $\implies$ is trivial. Conversely, suppose that $f$ is continuous on every compact subset of $S$. Let $p \in S$. Suppose $\{x_n\}$ is a sequence of points in $S$ converging to $p$. We claim $T = \{p, x_1, x_2, \ldots\}$ is a compact set. Indeed, let $\mathcal{F}$ be an open cover of $T$. Then there is some $F \in \mathcal{F}$ for which $p \in F$, and some $F_i \in \mathcal{F}$ such that $x_i \in F_i$. Let $\varepsilon > 0$ be so small that $B(p; \varepsilon) \subseteq F$. Then there is some $N > 0$ for which $d(p, x_n) < \varepsilon$ for each $n \geq N$. Therefore, $\{F, F_1, \ldots, F_N\} \subseteq \mathcal{F}$ is a finite subcover of $T$. Since $f$ is continuous on $T$ and $\{x_n\}$ is a sequence of points in $T$ converging to $p$, we must have $\lim_{n \to \infty} f(x_n) = f(p)$, which shows $f$ is continuous.

Exercise 4 (4.33) Give an example of a continuous $f$ and a Cauchy sequence $\{x_n\}$ in some metric space $S$ for which $\{f(x_n)\}$ is not a Cauchy sequence in $T$.

Proof. Consider the spaces $S = (0, 1)$, $T = (0, \infty)$ and the function $f(x) = \frac{1}{x}$. The sequence $x_n = 1/n$ is Cauchy in $S$, but the sequence $f(x_n) = n$ is certainly not Cauchy in $T$.

Exercise 5 (4.34) Prove that the interval $(-1, 1)$ in $\mathbb{R}^1$ is homeomorphic to $\mathbb{R}^1$. This shows that neither boundedness nor completeness is a topological property.

Proof. Take the function $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{1-x} - 1$ when $0 \leq x \leq 1$ and $f(x) = -f(-x)$ when $-1 \leq x \leq 0$. Then the function $g : \mathbb{R} \to (0, 1)$ defined by

$$g(x) = \begin{cases} 1 + \frac{1}{x+1} & x \geq 0 \\ -g(-x) & x \leq 0 \end{cases}$$
is a two sided inverse of $f$. Therefore, $f$ is a bijection. Further, both $f$ and $g$ are continuous away from 0 by the laws for continuity (e.g. the sum of continuous functions is continuous, etc.), and it is easily checked that $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0 = f(0) = g(0)$. Therefore, $f$ is a homeomorphism. ■