Math 205A
Partial solutions for homework assignment 8
Please email Josh Whitney at jwhitney {at} uci {dot} edu with any errors or oversights.

Exercise 1 (4.40) If \( x \) is a point in a metric space \( S \), let \( U (x) \) be the component of \( S \) containing \( x \). Prove that \( U (x) \) is closed in \( S \).

**Proof.** From exercise 4.39 which we proved in discussion, \( \overline{U} (x) \) is also a connected subset of \( S \) containing \( x \). Since \( U (x) \) is the union of all such subsets, \( U (x) \subset U (x) \) so \( U (x) \) is closed. ■

Exercise 2 (4.48) Let \( S \) be an open connected set in \( \mathbb{R}^n \). Let \( T \) be a component of \( \mathbb{R}^n - S \). Prove that \( \mathbb{R}^n - T \) is connected.

**Proof.** First we notice that

\[
\mathbb{R}^n - T = S \cup (\mathbb{R}^n - S - T) = S \cup \left( \bigcup_{x \in \mathbb{R}^n - S - T} U (x) \right)
\]

where each \( U (x) \) is the connected component of \( x \) in \( \mathbb{R}^n - S \) (the equality holds here because \( T \) is a component of \( \mathbb{R}^n - S \)). We claim that \( S \cap U (x) \) is nonempty for each \( x \in \mathbb{R}^n - S - T \). To see this, suppose \( S \cap U (x) \) is empty for some \( x \in \mathbb{R}^n - S - T \). Then \( U (x) \) is a closed subset of \( \mathbb{R}^n - S \) by exercise 4.40, and \( \mathbb{R}^n - S \) is closed in \( \mathbb{R}^n \) by the assumption that \( S \) is open. Therefore, \( U (x) \) is closed in \( \mathbb{R}^n \). On the other hand, \( \mathbb{R}^n - S \subset \mathbb{R}^n - S \), so \( U (x) \) must also be a component of \( \mathbb{R}^n - S \). Thus, \( U (x) \) is open in \( \mathbb{R}^n - S \) by the proof of theorem 4.44. Since \( \mathbb{R}^n - S \) is open in \( \mathbb{R}^n \), \( U (x) \) is open in \( \mathbb{R}^n \), contradicting that \( \mathbb{R}^n \) is connected and establishing the claim.

To finish the proof, let \( f \) be a two valued function on \( \mathbb{R}^n - T \) and let \( a, b \in \mathbb{R}^n - T \). We must show \( f (a) = f (b) \). If \( a \) and \( b \) are both in \( S \) then we are done by the connectedness of \( S \). Next, suppose \( a \in S \) but \( b \in U (x) \) for some \( x \in \mathbb{R}^n - S - T \). Let \( c \in U (x) \cap S \). Then \( f (c) = f (a) \) by continuity and \( f (c) = f (b) \) by the connectedness of \( U (x) \). If \( a \in U (x) \) and \( b \in U (y) \), then \( f (a) = f (c) = f (b) \) for any \( c \in S \) by what we just showed. ■

Exercise 3 (4.52) Assume that \( f \) is uniformly continuous on a bounded set \( S \) in \( \mathbb{R}^n \). Prove that \( f \) must be bounded on \( S \).

**Proof.** Since \( f \) is uniformly continuous on \( S \), there is some \( \delta > 0 \) for which \( |f (x) - f (y)| < 1 \) whenever \( x, y \in S \) and \( |x - y| < \delta \). Since \( S \) is bounded, \( S \) is contained in some closed ball \( B \) which, by compactness, we can cover with finitely many balls of radius \( \delta \). Therefore, we can cover \( S \) with finitely many balls of radius \( \delta \), say \( B_1, \ldots, B_n \). Assume that each \( B_i \) contains at least one point of \( S \) (if not, simply remove that ball from the list). Choose one point \( x_i \in S \) from each ball \( B_i \) arbitrarily and let \( M = \max_{i=1,\ldots,n} f (x_i) \). If \( x \in S \), then \( |x - x_i| < \delta \) for some \( 1 \leq i \leq n \) which implies

\[
|f (x) - f (x_i)| \leq 1.
\]

Hence, \( f (x) \leq 1 + f (x_i) \leq 1 + M \). Thus, \( |f (x)| \leq M + 1 \) for each \( x \in S \), completing the proof. ■

Exercise 4 (4.65) Let \( f \) be strictly increasing on a subset \( S \) of \( \mathbb{R} \). Assume that the image \( f (S) \) has one of the following properties: \( f (S) \) is open, \( f (S) \) is connected, or \( f (S) \) is closed. Prove that \( f \) must be continuous on \( S \).

Exercise 5 (4.67) Refer to exercise 4.66 and let \( C (S) \) denote the subset of \( B (S) \) consisting of all functions continuous and bounded on \( S \), where now \( S \) is a metric space.

1. Prove that \( C (S) \) is a closed subset of \( B (S) \).
2. Prove that the metric space \( C (S) \) is complete.

**Proof.**
1. Let $f$ be in the complement of $C(S)$ in $B(S)$. Then $f$ is not continuous at some point $s_0 \in S$. Therefore, there exists some $\varepsilon_0 > 0$ such that for each $\delta > 0$ there is some $s \in S$ with $d(s, s_0) < \delta$ but $|f(s) - f(s_0)| \geq \varepsilon_0$. Let $r = \varepsilon_0/4$ and let $g \in B(f;r)$. Then $|g(x) - f(x)| \leq \varepsilon_0/4$ for each $x \in S$. Let $\delta > 0$ be given. Then there is some $s \in S$ with $d(s, s_0) < \delta$ but $|f(s) - f(s_0)| \geq \varepsilon_0$. Thus, repeated application of the reverse triangle inequality gives

$$|g(s) - g(s_0)| \geq | |g(s_0) - f(s)| - |g(s) - f(s)||$$

$$\geq | |f(s_0) - f(s)| - |g(s_0) - f(s_0)| - |g(s) - f(s)|| \geq \varepsilon_0/2.$$ 

Therefore, $g$ is not continuous at $s_0$ and hence $g \in C(S)$. We have then shown that $B(f;r) \subset C(S)^C$ which means $C(S)$ is closed in $B(S)$.

2. We know that $C(S)$ is a closed subset of $B(S)$, the latter of which is complete by 4.66.b. The statement then follows from the fact that a closed subset of a complete metric space is complete.

Remark 6 The following question was posed to me after discussion today. Let $S = [0, 1]$ and take $f_n(x) = x^n$ for each $n \in \mathbb{Z}^+$. Then $f_n(x)$ is a sequence in $C(S)$. The potential problem here is that $f_n(x)$ converges pointwise to a function $f$ not in $C(S)$, which would contradict that $C(S)$ is closed as we showed above if $f_n \rightarrow f$ in the sup norm. Notice, however, that this is not the case. that while $f_n(x)$ converges pointwise to $f$, it does not converge to $f$ in the sup norm. Indeed, $\|f_n - f\| \geq 1/2$ for each $n \in \mathbb{Z}^+$, so while $f_n(x)$ converges pointwise to $f$, it does not converge to $f$ in the sup norm.