3. What is the order of \((1 \ 2)(3 \ 4 \ 5)(6 \ 7 \ 8 \ 9)\) in \(S_{11}\)?

This is a cycle decomposition (i.e., a product of disjoint cycles), so the order is \(\text{lcm}\{2, 3, 4\} = 12\).

4. What is the order of the element \(\overline{52}\) in the group \((\mathbb{Z}/64\mathbb{Z}, +)\)?

Since \(\overline{52}\) is \(52 \cdot -1\), and \(-1\) has order 64, the order of \(\overline{52}\) is \(\frac{64}{\gcd(64, 52)} = 64/4 = 16\).

5. For the (cyclic) group \((\mathbb{Z}/9000\mathbb{Z}, +)\), find the number of generators, and list two of them.

The generators are the elements relatively prime to 9000. Two of them are \(1\) and \(8999\) (which is \(-1\)). The number is \(\varphi(9000)\). Since \(9000 = 2^3 \cdot 3^2 \cdot 5^3\), we have \(\varphi(9000) = 2^2 (2-1) \cdot 3(3-1) \cdot 5^2 (5-1) = 4 \cdot 6 \cdot 25 \cdot 4 = 2400\).

6. Prove that if \(p\) is prime, \(n \in \mathbb{Z}^+\), \(G\) is a group, and \(|G| = p^n\), then \(|Z(G)| > 1\).

See class notes. Note that you need to state clearly that you are using the Class Equation, and what the \(g_i\)'s are in that.

7. Classify groups of order 6, up to isomorphism. (Give a proof.)

Note: This was homework exercise 10 on p.122 (which is before the Class Equation, so it’s not necessary to use that).

Answer: \(\mathbb{Z}_6\) and \(S_3\).

Proof: Note first that \(\mathbb{Z}_6\) and \(S_3\) are not isomorphic, since \(\mathbb{Z}_6\) is abelian while \(S_3\) is non-abelian.

Case 1: \(G\) is abelian. By Cauchy’s Theorem for abelian groups, \(G\) has an element \(x\) of order 2 and an element \(y\) of order 3. Let \(r\) denote the order of \(xy\). Then \(e = (xy)^r = x^r y^r\), so \(x^r = (y^{-1})^r \in \langle x \rangle \cap \langle y \rangle = \{e\}\) (since by Lagrange’s Theorem, \(|\langle x \rangle \cap \langle y \rangle|\) divides both 2 and 3, and thus is 1). Thus 2 divides
and 3 divides \( r \). So 6 divides \( r \). Since \( e = (xy)^6 \), we have \( |xy| = 6 \), so \( xy \) must generate \( G \), so \( G \) is cyclic and \( G \cong \mathbb{Z}_6 \).

(Instead of using Cauchy’s Theorem for abelian groups, you could have done the following. Suppose \( G \) is abelian and does not have an element of order 6. Then each of the (5) non-identity elements has order 2 or 3. If \( G \) has 2 elements of order 2, then they generate a Klein-4 subgroup of \( G \); but a group of order 6 can’t have a subgroup of order 4, by Lagrange; so \( G \) has at most 1 element of order 2. Since the elements of order 3 pair off (\( g \) pairs with \( g^{-1} = g^2 \)), we can’t have all 5 non-identity elements having order 3; at least (and thus exactly) one has order 2. Thus \( G \) has elements of orders 2 and 3.)

Case 2: \( G \) is not abelian. It follows from Lagrange’s Theorem that \( |G/Z(G)| = 1, 2, 3, \) or \( 6 \). If \( |G/Z(G)| = 1, 2, \) or \( 3 \), then \( G/Z(G) \) is cyclic, so \( G \) is abelian by a homework problem. Thus, \( |G/Z(G)| = 6 \), so \( Z(G) = \{e\} \).

Since \( G \) is not abelian, \( G \) is not cyclic, so all non-identity elements have order 2 or 3. Since the elements of order 3 pair off (\( g \) pairs with \( g^{-1} = g^2 \)), we can’t have all 5 non-identity elements of order 3; at least one, call it \( \sigma \), has order 2. Let \( H = \langle \sigma \rangle \). We first show \( H \) is not normal in \( G \).

If \( H \triangleleft G \), then for all \( g \in G \), we have \( g\sigma g^{-1} \in H = \{e, \sigma\} \).

Since \( \sigma \neq e \), we have \( g\sigma g^{-1} \neq e \), so \( g\sigma g^{-1} = \sigma \), so \( g\sigma = \sigma g \) for all \( g \in G \), so \( \sigma \in Z(G) \), contradicting that \( Z(G) = \{e\} \). Thus, \( H \) is not normal in \( G \).

Now \( G \) acts by left multiplication on the set \( A \) of left cosets of \( H \) in \( G \). Let \( \pi : G \to S_A \cong S_3 \) denote the associated permutation representation. Then \( \ker(\pi) \) is a normal subgroup of \( G \) contained in \( H \). Since \( |H| = 2 \) and \( H \) is not normal in \( G \), we must have \( \ker(\pi) = \{e\} \), so \( \ker(\pi) \) is injective. Since \( |G| = 6 = |S_3| \), \( \pi \) gives an isomorphism from \( G \) onto \( S_3 \), as desired.