Lecture 11: Overview

- Maximum Flow
Problem

- Have a directed graph
- Use to model flow of material from source to sink
- Edges contain maximum quantify of material that may flow across
- Use $c(u,v)$ to store capacity of edge from $u$ to $v$
Constraints

- Capacity constraint:
  - For all $u, v \in V$, $f(u,v) \leq c(u,v)$

- Skew Symmetry:
  - For all $u, v \in V$, $f(u,v) = -f(v, u)$

- Flow Conservation:
  - For all $u \in V - \{s,t\}$, we require
    \[
    \sum_{v \in V} f(u,v) = 0
    \]
Optimize

• Maximize the flow through the network

\[ |f| = \sum_{v \in V} f(s,v) \]
Multiple Sources/Sinks

- Can convert to single source/sink
- Idea:
  - Add supersource s and supersink t
  - Connect these to sources and sinks via edges with an infinite capacity
Ford-Fulkerson Method

FORD-FULKERSON-METHOD(G, s, t)
1 initialize flow f to 0
2 while there exist an augmenting path p
3   do augment flow f along p
4 return f
Residual Capacities

- Amount of additional flow we can push from \( u \) to \( v \) before exceeding the capacity \( c(u, v) \)
- \( c_f(u, v) = c(u, v) - f(u, v) \)
- Augmenting path is path through the residual network
Max flow Min-cut Theorem

• If $f$ is a flow in a flow network $G=(V, E)$ with source $s$ and sink $t$, then the following conditions are equivalent:
  1. $f$ is a maximum flow in $G$
  2. The residual network $G_f$ contains no augmenting paths
  3. $|f|=c(S, T)$ for some cut $(S,T)$ of $G$
What is the complexity of Ford-Fulkerson method?

- If the values are irrational and choices made poorly, it may not terminate!!
- Can even take a long time with integers:
Edmonds-Karp Algorithm

- Suppose we always choose the shortest path from $s$ to $t$ in the residual network?
- Complexity is $O(VE^2)$
- Why:
  - Each augmentation has complexity $O(E)$
  - Numbers of augmentations is $O(VE)$
    - Each edge can be a critical edge at most $|V|/2-1$ times
    - Once edge is critical, its flow is saturated
    - Must appear in path the other way next
    - Since previous path is shortest, this reversing path must be two edges longer
$d_f'(s,v) = d_f(s,u) + 1$

$d_f'(s,u) = d_f'(s,v) + 1$

$\geq d_f(s,v) + 1$ (paths increase monotonically)

$= d_f(s,u) + 2$
Maximum Bipartite Matching

- Idea: Have bipartite graph
- Want to pick maximum matching
- Matching is a subset of edges $M \subseteq E$ such that for all vertices $v \in V$ that at most one edge of $M$ is incident on $V$
- How is this related to maximal flow problem?
Idea:

- Add source and sink
- Connect source to all nodes on “left” side with edges of capacity 1
- Connect all nodes on “right” side to sink with edges of capacity 1
- Take all edges in original bipartite graph to have capacity 1