(1) Let $X$ be a random variable with density

$$f_X(x) = \begin{cases} 
  c + x & \text{if } -1 < x < 0 \\
  c & \text{if } 0 \leq x < 1 \\
  0 & \text{else}
\end{cases}$$

30 pts.

5 pts. (i) Find $c$.

10 pts. (ii) Find the C.D.F. of $X$.

5 pts. (iii) Give $P\left(-\frac{1}{2} < X < \frac{1}{2}\right)$.

5 pts. (iv) Find the C.D.F. for $Y = X^2$.

25 pts. (2) Assume that winter storms arrive "approximately" according to a Poisson process with a rate of 3 per month during the "storm season" November through March.

5 pts. (i) Give the probability of at least two storms in November.

10 pts. (ii) Give the probability of exactly 3 storms in November, 2 in December, and none in January, February, or March. Justify your answer.

10 pts. (iii) Suppose there were 10 storms in one season. What is the probability that exactly three of them were in November and exactly two in December?

20 pts. (3) Let $X \sim U[-1,1]$ and $Y \sim \text{exp}(1)$ and assume $X$ is distributed independently of $Y$.

5 pts. (i) Give the joint p.d.f. for $(X,Y)$.

10 pts. (ii) Find the C.D.F. for $Z = X + Y$.

15 pts. (4) Consider a machine with two major systems, say system $A$ and system $B$. System $A$ works only if parts $A_1$ and $A_2$ both work or if part $A_3$ works. System $B$ works only if parts $B_1$ and $B_2$ both work. The machine works only if systems $A$ and $B$ both work. Suppose parts $A_1$, $A_2$, and $A_3$ have lifetimes which are exponential ($\lambda_A = 2/\text{year}$) and $B_1$ and $B_2$ have lifetimes which are $\text{exp}(\lambda_B = 1/\text{year})$. Assume mutual independence of all parts.

15 pts. (i) Find the p.d.f. for the time until breakdown of the machine.

10 pts. (ii) Find the probability that system $A$ fails before system $B$ fails.
1. Suppose people's weights are normally distributed with mean $\mu = 150$ pounds and variance $\sigma^2 = (50)^2$ (pounds$^2$).

(a) Give an appropriate sketch of the density.

(b) Find the probability that a randomly selected person weighs between 100 and 200 pounds.

(c) How much must you weigh so that only 1% of all people are heavier than you are?

(d) If sixteen randomly selected people step into an elevator which has a maximum capacity of 2600 pounds, what are the chances the elevator is overloaded? Justify.

(e) If 100 people are randomly selected, what are the chances that 25 weigh less than 100 pounds, 50 weigh between 100 and 200 pounds and 25 weigh over 200 pounds?
2. Suppose you have five enemies, each of whom could harm you at any moment in time. Suppose your enemies behave independently of one another, and that the waiting times until they harm you are each exponential with parameter \( \lambda = 1 \) per year. Let \( L_1 \) denote the time until you are harmed by any of these individuals.

(a) Find \( P(L_1 > t) \) for \( t > 0 \). Identify the distribution of \( L_1 \). Hint: Note that \( [L_1 > t] \) occurs if and only if all your enemies wait until after time \( t \) to harm you.

Now let \( L_5 \) be the waiting time until you are harmed for the 5th time.

(b) Identify the distribution of \( L_5 \). Give \( EL_5 \). Hint: You may assume that harm comes to you according to a Poisson Process.
6. Suppose two fair dice are rolled until a 7 or 11 occurs. Let $X =$ the number of rolls.

(a) What is the distribution of $X$? Give $EX$.

The dice are rolled independently until a 7 or 11 occurs for the 100th time. Let $Y$ be the total number of rolls until this event.

(b) What is the distribution of $Y$? Give $EY$.

(c) Give the approximate distribution of $Y$. *Justify.* You may use the fact that $Var(X) = 14$.

(d) Give the approximate probability that $Y$ does not exceed 500 rolls.

(e) If the dice are rolled independently 10 times, what are the chances that a 7 or 11 comes up 2 or fewer times.
8. Let $X \sim \text{Poisson}(\lambda)$.

(a) Show that $E X^2 = \lambda^2 + \lambda$, and give $\text{Var}(X)$.

(b) If $\lambda = 100$, say, how could you justify the following approximation?

$$P(X < 120) = \sum_{k=0}^{120} \frac{(100)^k e^{-100}}{k!} = \int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz = .9728$$

HINT: Use the Central Limit Theorem and other appropriate results.
A subcommittee of size 4 is to be selected from the Senate armed services committee which has 10 members; 4 are democrats and 6 are republicans. Among the democrats, 3 are liberal and one is moderate, and among the republicans, 4 are conservative and 2 are moderate. **Do not do arithmetic for this problem.** Assuming the committee is selected at random, what are the chances that:

5. (a) all are republicans?

5. (b) all are conservatives or all are republicans?

5. (c) there is at least one republican?
[5] (d) there are exactly two conservatives and exactly one democrat?

Now suppose the committee is to have a chair and a vice chair. What are the chances that

[10] (e) the chair is conservative, the vice chair is liberal and the rest are moderate?
(2) Suppose it is known that a lie detector will correctly detect a lie 80% of the time, and that it correctly says a person is not lying, when they are telling the truth, 70% of the time. Furthermore suppose it is known that 20% of the people that are given lie detector tests are actually lying.

[10] (a) What proportion of the time does the test indicate that a lie is being told? Justify.

[10] (b) If the test indicates the individual is lying, what are the chances that they are actually lying? Comment on the result.
(3) Suppose that events A, B, and C are mutually independent events and that $Pr(A) = Pr(B) = Pr(C) = p$ for some value $p$.

[8] (a) Calculate $Pr(AB \cup C)$ in terms of $p$.

[7] (b) Derive $Pr(A|B \cup C)$ in terms of $p$. Show steps.

[5] (c) Calculate $Pr(B \cup C|AC)$ in terms of $p$. 
(4) Suppose that individual shrubs die within one year of planting with probability 0.2, and that they die independently of one another. Suppose you have just planted 25 shrubs in your yard. **Do not do arithmetic for this problem**

[8] (a) What is the probability that none of your shrubs die within the first year? **Justify.**

[7] (b) What are the chances that exactly 15 shrubs die within one year?

[5] (c) What are the chances that at least one shrub dies within one year?
Now suppose that individual plants that survive for one year after planting have one chance in 50 of dying (independently of one another) in the second year after planting.

[10] (d) What are the chances that exactly 3 plants, of the original 25, die during year two? Hint: This is a bit tricky.

(5) Suppose 8 dogs are to be randomly placed into 4 pens with exactly two dogs in a pen. Suppose exactly 4 of the dogs have fleas.

[10] What are the chances that, after the dogs are placed, there will be exactly 6 dogs with fleas? Be sure to explain yourself for maximum partial credit.
1. The U.S. Senate consists of 100 members including 40 republicans and 60 democrats. Assume there are 5 radical conservative, 30 regular conservative, and 5 moderate republicans; and that there are 2 radical liberal, 40 regular liberal, and 18 moderate democrats. A committee of 10 is to be formed based on volunteers to re-evaluate the process of appointing justices to the Supreme Court. Assume the committee must consist of 5 republicans and 5 democrats and that all such committees are equally likely.

Points

(a) How many elements are there in the sample space? (Give an example of a single element.)

\[
\binom{60}{5} \binom{40}{5} \text{ elements, } \{a_1, a_2, \ldots, a_{18}, b_1, b_2, \ldots, b_{18}\}
\]

(b) Give the probability all the radical-conservative senators are on the committee.

\[
\binom{5}{2} \left(\frac{58}{3}\right) / \binom{60}{5} \binom{40}{5}
\]

(All 5 RC) (Both RC & Dem) (3 non Dem)

(c) Give the probability that at least one radical-conservative senator is on the committee.

\[
1 - P(\text{none}) = 1 - \frac{\binom{30}{5} \left(\frac{58}{3}\right)}{\binom{60}{5} \binom{40}{5}}
\]

(5 non RC) (58 RC)

(5 non RC) (5 RC)
(d) Give the probability there are 2 hard-ball senators and 8 moderates.

\[
\binom{5}{2} \binom{5}{3} \binom{60}{5} + \binom{5}{1} \binom{5}{4} \binom{18}{1} \binom{5}{2} + \binom{5}{4} \binom{5}{3}
\]

Now suppose Senator K, a democrat, will not serve on the committee with senator S, a republican.

(e) How many committees are possible now?

\[
\binom{59}{5} \binom{59}{5} + \binom{39}{4} \binom{5}{5} + \binom{39}{5} \binom{5}{3}
\]

or \( \binom{60}{5} \binom{40}{5} - \binom{59}{3} \binom{39}{2} \binom{2}{2} \)
Points

[20]

2. Suppose 30% of all working women are "sexually harassed" on the job at some point in their careers. Further suppose that 93% of such abuses go unreported. Finally, suppose that 1% of working women who have not been harassed, falsely claim sexual harassment when asked. A woman has just made an accusation of sexual harassment.

What are the chances, under the above assumptions, that she has actually been harassed?

Hint: Let R denote "reported" and H denote "sexual harassment".

\[ P(H | R) = \frac{P(R | H) P(H)}{P(R)} \]
\[ = \frac{(0.30)(0.07)}{(0.30)(0.07) + (0.7)(0.01)} \]
\[ = 0.75 \]

(b) What is the proportion of women, among those not filing complaints of sexual harassment, that have in fact been harassed?

\[ P(H | R^c) = \frac{P(R^c | H) P(H)}{P(R^c)} \]
\[ = \frac{(0.98)(0.30)}{(0.98)(0.30) + (0.02)(0.30)} \]
\[ = 0.972 \]
3. A single fair 6 sided die is rolled (independently) 10 times.

Points

[5] (a) How many possible outcomes are there? Describe the sample space. 

\[ S = \{6^{10}\} \text{ tuples like } (1,1,\ldots,1), \ldots, (6,6,\ldots,6) \]

[5] (b) Give the probability that the first 5 rolls are ones and the second 5 are odd.

\[ P(1,1,1,1,1,0,0,0,0,0) = \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^5 \]

[5] (c) Give the probability of exactly 5 ones.

\[ \binom{10}{5} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^5 \]
(d) Give the probability of exactly 5 \textit{ones}, 1 \textit{three} and 2 \textit{evens}.

\[
\binom{10}{5,1,2,2} \cdot \frac{1}{5} \cdot 1 \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{10}{15} = \frac{2}{3}
\]

Find plans for 5 ones, 1 three, 2 evens, etc.

[5] (e) Give the probability of all even or at least one odd.

\[
1 = P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

\[
= \left(\frac{1}{2}\right)^6 + 1 - \left(\frac{1}{2}\right)^6 = 0
\]
4. Suppose there are 100 houses on the market in your town, and that you set out to look at them, at random, until you find a "suitable" home. You will _not_ return to houses that you already looked at, and you will stop once you have found one that is "suitable". Suppose there are only 5 houses that are "suitable".

**Points**

[5] (a) What is the probability you find a suitable house on the second try if you failed on the first?

\[ P(S_2 | F_1) = \frac{5}{99} \]

[5] (b) Derive the probability that you find a suitable house on the second try.

\[ P(S_2) = P(F_1 S_2) = P(F_1) P(S_2 | F_1) = \frac{95}{100} \cdot \frac{5}{99} \]

Now suppose that only 2 of the 5 suitable houses are "perfect" and suppose that you will only buy a "perfect" house.

[10] (c) What are the chances that you only find a "perfect" house after finding two suitable but not perfect houses? Hint: The answer is easy if you choose the right sample space.

\[ S = \{ \text{arrangements of 2 P's and 3 S's} \} \]

\[ \binom{5}{2} \]

\[ P(SSPPSP \text{ or } SSSPPS) = \frac{2}{\binom{5}{2}} = \frac{2}{10} = \frac{1}{5} \]
Points

[5] 5. Show that

\[ P(A \cap B) = P(A)P(B) \]

implies that

\[ P(A | B) = \frac{P(A \cap B)}{P(B)} \]

[5] 2. Argue that \( A \cap B = \emptyset \) does not imply that \( A \) is independent of \( B \).

Ex: Toss a coin twice

\[ A = \text{H} \quad B = \text{T} \]

\[ \Rightarrow A \cap B = \emptyset \]

But

\[ \frac{1}{2} = P(A) \neq P(A | B) = 0 \]

\[ \Rightarrow A \perp B \]
Instructions: This is an open book and open notes bluebook examination. Show all work. Do not do arithmetic. Try all problems. Do easy problems first. Good Luck!

On the back of the bluebook, write:

"I hereby agree to have my grade posted by my student I.D. number."

and sign your name if you want me to post grades.

Points 20

1. Consider the following p.d.f.'s.

(i) \[ \frac{1}{2} \]

(ii) \[ \frac{1}{2} \]

(iii) \[ \frac{1}{2} \]

(iv) \[ \frac{1}{2} \]

(v) \[ \frac{1}{2} \]

Suppose you have observed 100 observations on a R.V. which has one of the above densities. Which density or densities would you guess to be appropriate if:

a. (3 points) 50 observations fell between \(-1/2\) and \(1/2\)?

b. (3 points) 25 observations fell between \(-1/2\) and \(1/2\)?

c. (3 points) 25 observations between \(-1\) and \(0\)?

d. (3 points) 50 observations between \(0\) and \(1\)?

e. (4 points) Is the variance in (i) \(\geq\), \(\leq\), or \(=\), to the variance in (ii)? Why?

f. (4 points) Is the variance in (iv) \(\geq\), \(\leq\), or \(=\), to the variance in (v)? Why?

EXAM CONTINUED ON PAGE 2
20. Suppose people's weights are distributed as $N(150 \text{ pounds}, (50)^2 \text{ (pounds)}^2)$.  
   a. (3 points) Give the probability that a randomly selected person weighs more than 200 pounds.
   b. (7 points) If ten people are randomly selected and weighed, what are the chances that two or more weigh over 200 pounds?
   c. (3 points) If people are randomly selected and weighed one at a time, how many people do you expect to weigh before finding three people who weigh over 200 pounds?
   d. (7 points) Sixteen people step onto an elevator which has a capacity of 2600 pounds. What are the chances the elevator is overloaded?

20. (10 points) Explain the relationships among the following R.V.'s.
   i. Bernoulli ($p$).
   ii. Binomial ($n, p$).
   iii. Hypergeometric ($N, M, n$).
   iv. Negative Binomial ($r, p$).
   v. Geometric ($p$).
   b. (7 points) Explain the relationships among:
      vi. Exponential ($\lambda$).
      vii. $\Gamma(k, \lambda)$.
      viii. Poisson ($\lambda$).
   c. (3 points) What analogies can you make between (v) and (vi) and between (iv) and (vii)?

20. Let $X$ and $Y$ be i.i.d. $U[0,2]$ random variables.
   a. (6 points) Find $EX$, $Var(X)$.
   b. (5 points) Find $E(X + Y)$, $Var(X + Y)$.
   c. (7 points) Find $cov(X - Y, X)$.
   d. (7 points) Find $P(X + Y \leq 3)$.

20. Suppose it is known that 30% of all people who go to the beach consume alcohol while there. Further, assume that 60% of the drinkers and 50% of the non-drinkers go swimming.
   a. (7 points) What proportion of the people who go to the beach go swimming?
   b. (7 points) What are the chances that a swimmer has been drinking?
   Now, suppose that drinking swimmers have a 1/100 chance of drowning while non-drinking swimmers have a 1/200 chance of drowning.
   c. (6 points) What are the chances a swimmer will drown?