Suppose there are \( n \times m \) distinct objects and you are to select \( r \) of them to form a group. Moreover, suppose no of the objects are of one color and \( m \) of another. The number of groups, without taking color into account, is \( (n+m)^r \). Observe that the number of possibilities with exactly \( j \) of one color (i.e., \( r-j \) of the other), is \( \binom{n}{j} \binom{m}{r-j} \). Since there must be exactly 0 or exactly one or ... or exactly \( j \) or ... or exactly \( r \) of the first color. The total number of ways without regard to color must be

\[
\sum_{j=0}^{r} \binom{n}{j} \binom{m}{r-j}
\]
19 (a) we have \( \binom{5}{3} \) committees of women and \( \binom{6}{3} \) possible groups of men where \( \binom{3}{4} \) groups have 2 men who cannot work together. So we have \( \binom{5}{3} \left[ \binom{6}{3} - 4 \right] = 56 \times 16 = 896 \) possible committees.

(b) As above but with \( \binom{6}{3} - \binom{3}{2} \binom{3}{6} \) groups of women and \( \binom{5}{3} \) groups of men. So \( (56-6)(20) = 1000 \) committees.

(c) We have \( \binom{5}{2}\binom{9}{9} \) possible committees but \( \binom{1}{2} \binom{3}{1} \binom{1}{3} \) groups that we cannot have, so we have \( 56 \times 20 - (21 \times 10) = 910 \) possible committees.

20 (a) we have \( \binom{5}{2} \) possible groups, of which \( \binom{3}{2} \binom{3}{5} = 20 \) groups cannot occur. So we have 56 - 20 = 36 possible parties.

(b) we have \( \binom{6}{3} \) groups of people with neither friend, and \( \binom{3}{3} \) groups where both attend. So we have \( \binom{6}{3} + \binom{3}{3} = 6 + 20 = 26 \) possible parties.
35. \( P(\text{at least 1 psychologist}) = 1 - P(\text{no psychologist}) \)

\[
= 1 - \frac{30}{54} \cdot \frac{26}{53} \cdot \frac{25}{52} = .8363
\]

\* 41. \( P(\text{at least 1 6}) = 1 - P(\text{no 6s}) \)

\[
= 1 - \frac{54}{56} \cdot \frac{55}{56} \cdot \frac{56}{56} = .5177
\]

\* 42. In \( n \) rolls, \( P(\text{at least 1 double 6}) = 1 - P(\text{no 6s}) \)

\[
= 1 - \left(\frac{35}{36}\right)^n
\]

so at what \( n \) is \( P(\text{at least 1 pair of 6s}) \)?

\[
\left(\frac{36}{36}\right)^n = .5 \quad \text{or} \quad \log\left(\frac{36}{36}\right) = 10 \log .5
\]

gives us \( n = 25 \)

46. \( P(\text{at least 1 pair matching months}) = 1 - P(\text{no match}) \)

\[
= 1 - \frac{\binom{12}{1}}{12^n} \quad \text{so} \quad \frac{\binom{12}{1}}{12^n(12-n)!} = .5
\]

and solve for \( n \).
Pr ( 2 matching days) = 1 - Pr (None) = 1 - \frac{12 \times 11 \times \ldots \times (12-N+1)}{12^N}

Find smallest N for which this is true. Solution is by trial and error.

N = 2 \implies Pr = \frac{1}{2}
N = 3 \implies Pr = 1 - \frac{11 \times 10}{12^2} = \frac{34}{744} < \frac{1}{2}
N = 4 \implies Pr = 1 - \frac{11 \times 10 \times 9}{12^3} = 1 - \frac{3310}{744} = 1 - \frac{2774}{48} = \frac{14}{48} > \frac{1}{2}
N = 5 \implies Pr = 1 - \frac{11 \times 10 \times 9 \times 8}{12^4} = 1 - \frac{2774}{48} > \frac{1}{2}

So, N = 5 works.

P. 60 \pm 52

(9) Pr (no complete pair) = \frac{\binom{10}{8}}{\binom{20}{8}}

(10) Pr (exactly 1 complete pair) = \binom{10}{2} \binom{2}{1} \binom{9}{6} \binom{12}{2}

Pick 6 pairs from remaining 9
Pick exactly 1 pair
Pick both from one pair
Pick each from one pair
Pick exactly 1 pair
Pick one from each pair
7. The king cannot have an older brother; he can have a younger brother, an older sister, or a younger sister.

\[ \therefore P(\text{sister}) = \frac{2}{3}. \]

12. (a) \[ A = \text{pass exam 1} \]
\[ B = \text{pass exam 2} \]
\[ C = \text{pass exam 3} \]

\[ P(C|AB)P(B|A)P(A) = P(ABC) \]
\[ P(ABC) = 0.9 \times 0.8 \times 0.7 = 0.504 \]

(b) \[ P(B^c|A) = \frac{P(AB^c)}{P(A)} = \frac{0.9 \times 0.2}{0.9} = 0.2 \]

15. \( E = \) ectopic pregnancy \( S = \) smoker

\[ P(E|S) = \frac{\text{P(E) given } S = 0.33}{1} \]

What is \( P(S|E)? \)
\* P. 112 \pm 15

Given

\[ P_r(E|S) = 2 \times P_r(E|\text{NS}) \]

\[ P_r(S) = 0.32 \]

\[ \Rightarrow \frac{P_r(E|\text{NS})}{P_r(E|S)} = \frac{1}{2} \]

Find \[ P_r(S|E) \]

\[ P_r(S|E) = \frac{P_r(E|S) \cdot P_r(S)}{P_r(E|S) \cdot P_r(S) + P_r(E|\text{NS}) \cdot P_r(\text{NS})} \]

\[ = \frac{0.32}{0.32 + \frac{P_r(E|\text{NS})}{P_r(E|S)} \cdot 0.68} \]

\[ = \frac{0.32}{0.32 + \frac{0.68}{0.32}} \]

\[ = \frac{0.32}{0.32 + \frac{1}{2} \cdot 0.68} \]

\[ = \frac{0.32}{0.32 + 0.34} \]

\[ = \frac{0.32}{0.66} = \frac{16}{33} \]
So in 100 cakes, 50 are cooked by A with $0.02 \times 50 = 1$ failure.

Of 100 cakes, $50 \times 0.02 + 30 \times 0.03 + 20 \times 0.07 = 2.9$ are failures, so A causes $0.34$ of the failures.

In a year, $0.05 \times 1 + 0.15 \times 5 + 0.3 \times 3 = 1.75$, so 17.5% of the population has an accident in any given year.

P 118 # 60

(a) Since his son has blue eyes, both parents must have brown, blue genes.

\[
\begin{array}{c|c|c}
 & \text{Brown} & \text{Blue} \\
\hline
\text{Brown} & B & B \\
\text{Brown} & B & L \\
\text{Blue} & L & L \\
\end{array}
\]

So $P(\text{Smith has a blue gene}) = \frac{2}{3}$.

(b) Chance blue is passed on:

\[
P(\text{Brown, Brown}) = \frac{1}{3} \cdot 0 + \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{2}
\]

Chance brown is passed on:

\[
P(\text{Brown, Blue}) = \frac{1}{3} \cdot 0 + \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{2}
\]
Solution to problem 61

At beginning of time, first child must be Aa or AA since it is normal. We must have \( P(Aa\, |\, \text{normal}) = \frac{2}{3} \) = 1 - \( P(AA\, |\, \text{normal}) \) since then would be \( \frac{1}{4} \) prob of each of \( \{AA, Aa, Aa, aa\} \) contributing.

(4) The mate of this child (after he/she grows up a little hopefully) is a carrier so they are Aa. If the new mom and dad are AA and Aa then the prob of aa is zero. If the new mom and dad are Aa and Aa then the prob of aa is \( \frac{1}{4} \). So

\[
Pr(\text{aa, background}) = Pr(\text{aa, BI}, \{\text{Aa, Aa}\}^* \cup \text{parents are parents of Aa, Aa}) \\
= Pr\left(\frac{1}{4}\right) Pr\left(\text{1st child | BI}\right) Pr\left(\text{mate is | BI}\right) \\
= \left(\frac{1}{4}\right) \frac{2}{3} \cdot 1 = \frac{2}{12} = \frac{1}{6}
\]

(6) \( Pr(\text{second child | 1st child is AA, BI}) \) = \( Pr((aa)_2 | (aa)_1^c, BI) \)

\[
= Pr((aa)_2, (aa)_1^c | BI) / Pr((aa)_1^c | BI) \\
= Pr((aa)_2, (Aa), V(AA), | BI) / (1 - \frac{1}{6}) \text{ using part (5)} \\
= \frac{1}{6} \left[ Pr((aa)_2, (Aa), | BI) + Pr((aa)_2, (AA), | BI) \right]
\]
But

$$\Pr\left( (AA, aA) \mid B, I \right) =$$

$$\Pr\left( (AA, aA) \mid (AA, AA), B, I \right) \Pr\left( (AA, AA) \mid B, I \right)$$

$$+ \Pr\left( (AA, aA) \mid (aa, AA), B, I \right) \Pr\left( (aa, AA) \mid B, I \right)$$

$$+ \Pr\left( (AA, aA) \mid (aa, aA), B, I \right) \Pr\left( (aa, aA) \mid B, I \right)$$

$$= 0 + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{12}$$

Next we calculate

$$\Pr\left( (AA, aA) \mid B, I \right) =$$

$$\Pr\left( (AA, aA) \mid (AA, AA), B, I \right) \Pr\left( (AA, AA) \mid B, I \right)$$

$$+ \Pr\left( (AA, aA) \mid (aa, AA), B, I \right) \Pr\left( (aa, AA) \mid B, I \right)$$

$$+ \Pr\left( (AA, aA) \mid (aa, aA), B, I \right) \Pr\left( (aa, aA) \mid B, I \right)$$

$$= \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{24}$$

So

$$\Pr\left( \text{second child is } aa \mid \text{1st is } aA \right) =$$

$$\left( \frac{1}{12} + \frac{1}{24} \right) / \frac{5}{6} = \frac{3/24}{5/6} = \frac{6}{40} = \frac{3}{20}$$
Chapter 2

(a) Consider the question in two steps.

Firstly, since there is no complete pair for eight shoes we select, these eight shoes must come from eight different pairs. So we select eight pairs out of 10 pairs. This can be done in \( \binom{10}{8} \) ways.

Secondly, for the eight pairs of shoes we selected, we can take either left or right for each pair, that is \( 2^8 \) ways to select eight shoes from the eight pairs with one shoe per pair.

So combine them, we have \( \binom{10}{8} \cdot 2^8 \) ways to select eight shoes from 10 pairs with no complete pair.

The total number of ways to select 8 shoes out of 10 pairs (20 shoes) is \( \binom{20}{8} \)

So \( P(\text{no complete pair}) = \frac{\binom{10}{8} \cdot 2^8}{\binom{20}{8}} \)

(b) Similar idea as the (1st one)

\( 0 \) Since exactly 1 pair is selected, we choose 1 pair out of 10 pairs, that is \( \binom{10}{1} \)

\( 0 \) Now we have 9 pairs left and we need to select 6 shoes from them with no complete pair. In terms of part (a), the same idea; it is \( \binom{9}{6} \cdot 2^6 \)

\( 3 \) again total possible ways are \( \binom{20}{8} \),

\[ P(\text{exactly 1 pair}) = \frac{\left(\binom{10}{1} \cdot 6 \right) \cdot 2^6}{\binom{20}{8}} \]
Problem 21

We are given the following joint probabilities

\[ P(W_<, H_<) = \frac{212}{500} = 0.424 \]
\[ P(W_<, H_> = \frac{198}{500} = 0.396 \]
\[ P(W_>, H_<) = \frac{36}{500} = 0.072 \]
\[ P(W_>, H_> = \frac{54}{500} = 0.108 . \]

Where the notation \( W_< \) is the event that the wife makes less than 25,000, \( W_> \) is the event that the wife makes more than 25,000, \( H_< \) and \( H_> \) are the events that the husband makes less than or more than 25,000 respectively.

Part (a): We desire to compute \( P(H_<) \), which we can do by considering all possible situations involving the wife. We have

\[ P(H_<) = P(H_<, W_<) + P(H_<, W_> = \frac{212}{500} + \frac{36}{500} = 0.496 . \]

Part (b): We desire to compute \( P(W_>, H_> \) which we do by remembering the definition of conditional probability. We have \( P(W_>, H_> = \frac{P(W_>, H_>)}{P(H_>)} \). Since \( P(H_> = 1 - P(H_<) = 1 - 0.496 = 0.504 \) using the above we find that \( P(W_>, H_> = 0.2142 = \frac{4}{11} . \)

Part (c): We have

\[ P(W_>, H_<) = \frac{P(W_>, H_<))}{P(H_<)} = \frac{0.072}{0.496} = 0.145 = \frac{9}{62} . \]

Problem 42 (special cakes)

Let \( R \) be the event that the special cake will rise correctly. Then from the problem statement we are told that \( P(R|A) = 0.98, P(R|B) = 0.97, \) and \( P(R|C) = 0.95, \) with the prior information of \( P(A) = 0.5, P(B) = 0.3, \) and \( P(C) = 0.2. \) Then this problem asks for \( P(A|R^c). \) Using Bayes’ rule we have

\[ P(A|R^c) = \frac{P(R^c|A)P(A)}{P(R^c)} , \]

where \( P(R^c) \) is given by conditioning on \( A, B, \) or \( C \) as

\[ P(R^c) = P(R^c|A)P(A) + P(R^c|B)P(B) + P(R^c|C)P(C) \]
\[ = 0.02(0.5) + 0.03(0.3) + 0.05(0.2) = 0.029 , \]

so that \( P(A|R^c) \) is given by

\[ P(A|R^c) = \frac{0.02(0.5)}{0.029} = 0.344 . \]
Theoretical problems

Problem 3 (biased selection of the first born)

We define $n_1$ to be the number of families with one child, $n_2$ the number of families with two children, and in general $n_k$ to be the number of families with $k$ children. In this problem we want to compare two different methods for selecting children. In the first method, $M_1$, we pick one of the $m$ families and then randomly choose a child from that family. In the second method, $M_2$, we directly pick one of the $\sum_{i=1}^{k} \hat{n}_i$ children randomly. Let $E$ be the event that a first born child is chosen. Then the question seeks to prove that

$$P(E|M_1) > P(E|M_2).$$

We will solve this problem by conditioning no the number of families with $i$ children. For example under $M_1$ we have (dropping the conditioning on $M_1$ for notational simplicity) that

$$P(E) = \sum_{i=1}^{k} P(E|F_i)P(F_i),$$

where $F_i$ is the event that the chosen family has $i$ children. This later probability is given by

$$P(F_i) = \frac{n_i}{m},$$

for we have $n_i$ families with $i$ children from $m$ total families. Also

$$P(E|F_i) = \frac{1}{i},$$

since the event $F_i$ means that our chosen family has $i$ children and the event $E$ means that we select the first born, which can be done in $\frac{1}{i}$ ways. In total then we have under $M_1$ the following for $P(E)$

$$P(E) = \sum_{i=1}^{k} P(E|F_i)P(F_i) = \sum_{i=1}^{k} \frac{1}{i} \left( \frac{n_i}{m} \right) = \frac{1}{m} \sum_{i=1}^{k} \frac{n_i}{i}. $$

Now under the second method again $P(E) = \sum_{i=1}^{k} P(E|F_i)P(F_i)$ but under the second method $P(F_i)$ is the probability we have selected a family with $i$ children and is given by

$$\frac{\hat{n}_i}{\sum_{i=1}^{k} \hat{n}_i},$$

since $\hat{n}_i$ is the number of children from families with $i$ children and the denominator is the total number of children. Now $P(E|F_i)$ is still the probability of having selected a family with $i$th children we select the first born child. This is

$$\frac{n_i}{\hat{n}_i} = \frac{1}{i},$$

since we have $\hat{n}_i$ total children from the families with $i$ children and $n_i$ of them are first born. Thus under the second method we have

$$P(E) = \sum_{i=1}^{k} \left( \frac{1}{i} \right) \left( \frac{\hat{n}_i}{\sum_{i=1}^{k} \hat{n}_i} \right) = \frac{1}{\left( \sum_{i=1}^{k} \hat{n}_i \right)} \sum_{i=1}^{k} \frac{n_i}{i}. $$
Then our claim that \( P(E|M_1) > P(E|M_2) \) is equivalent to the statement that

\[
\frac{1}{m} \sum_{i=1}^{k} \frac{n_i}{i} > \frac{\sum_{i=1}^{k} n_i}{\sum_{i=1}^{k} i n_i}
\]

or remembering that \( m = \sum_{i=1}^{k} n_i \) that

\[
\left( \sum_{i=1}^{k} i n_i \right) \left( \sum_{j=1}^{k} \frac{n_j}{j} \right) \geq \left( \sum_{i=1}^{k} n_i \right) \left( \sum_{j=1}^{k} n_j \right)
\]

But

\[
\frac{k}{\sum_{i=1}^{k} i n_i} > k \frac{k}{\sum_{i=1}^{k} n_i} \quad \text{and} \quad \frac{k}{\sum_{j=1}^{k} j} > \frac{k}{\sum_{j=1}^{k} n_j}
\]

Then

\[
\left( \frac{1}{\sum_{i=1}^{k} i n_i} \right) \left( \frac{k}{\sum_{i=1}^{k} n_i} \right) \frac{k}{\sum_{j=1}^{k} j} \frac{k}{\sum_{j=1}^{k} n_j} > \frac{k}{\sum_{i=1}^{k} n_i} \frac{k}{\sum_{j=1}^{k} n_j} \frac{k}{\sum_{j=1}^{k} n_j} \\
= \frac{(k}{\sum_{i=1}^{k} n_i} \right)^2
\]