1. (20) Suppose that mammograms indicate breast cancer 80 percent of the time when it is there to detect, and that they falsely indicate breast cancer 10 percent of the time when it is not there to detect. Also suppose that one percent of women who are tested for breast cancer actually have it.

(a) (7) What proportion of women test positive for breast cancer?

\[
P(BC^+) = \frac{TP}{TP + FP} = \frac{P(BC^+|C) + P(BC^+|\bar{C})}{P(C)P(BC^+|C) + P(\bar{C})P(BC^+|\bar{C})} = \frac{1}{100} \frac{8}{10} + \frac{99}{100} \frac{1}{100} = \frac{107}{1000}
\]

(b) (8) Among those women who test negative for breast cancer, what proportion actually have it? Do the arithmetic.

\[
P(C|BC^-) = \frac{P(BC^-|C)P(C)}{P(BC^-)} = \frac{\frac{2}{100} \frac{9}{10}}{1 - \frac{107}{1000}} = \frac{2}{893}
\]

(c) (7) If 10 women just tested negative for breast cancer, what is the probability that one or more of them will actually have it? Justify. No arithmetic.

Let \(X = \# \text{ who test neg.}\)

\[X \sim \text{Bin}(10, \frac{2}{100})\]

\[
Pr(X \geq 1) = 1 - Pr(X = 0) = 1 - \left(1 - \frac{891}{893}\right)^{10}
\]

\[= 1 - \left(\frac{891}{893}\right)^{10}
\]

\[\text{Note: } Pr(C|BC^-) = 1 - Pr(C|BC^+) = 1 - \frac{2}{893}\]
2d Method (ii)

\[ 1 - P(\text{both } L_1 \text{ and } C_1 \text{ are chair or VC}) \]

\[ = 1 - \frac{\binom{13}{3} \times \binom{1}{1}}{\binom{12}{3} \times \binom{1}{1}} = 1 - \frac{65}{66} = \frac{65}{66} \]

2. (25) Suppose a subcommittee of size 4 of the city council is to be formed. There are 12 people on the city council, of whom 4 are liberals, 5 are conservatives and 3 are independents. One of the members of the subcommittee will be selected as chair, another as vice chair, and the two remaining will just be regular members. Assume all selections are completely random. No arithmetic.

(a) (5) What is the probability that all members of the subcommittee are conservatives?

\[ P(\text{all Cons}) = \frac{\binom{4}{4}}{\binom{12}{4}} = \frac{5 \times 3}{12 \times 11 \times 10} \]

(b) (5) What is the probability that the chair and vice chair are conservatives and that the regular members are not.

(c) (5) What is the probability that the chair is a conservative, the vice chair a liberal, and the regular members are independents?

\[ \frac{\binom{4}{2}}{\binom{12}{2}} \]

(d) (10) A particular liberal and a particular conservative despise each other. What is the probability that they will not serve together in the leadership of the subcommittee.

Let \( L_1 \) and \( C_1 \) be the 2 members who despise each other.

\[ P(L_1 \text{ is not leader}, \text{ and } C_1 \text{ is not leader}) + P(C_1 \text{ is not leader}, \text{ and } L_1 \text{ is not leader}) + P(\text{neither } L_1 \text{ nor } C_1 \text{ is chair or VC}) \]

\[ = \frac{(1)(10)(10)}{12 \times 11} + \frac{(10)(1)(1)}{12 \times 11} + \frac{(10)(10)}{12 \times 11} \]

\[ = \frac{20}{12 \times 11} + \frac{20}{12 \times 11} + \frac{10}{12 \times 11} = \frac{65}{66} \] (Method II see top)
3. (26) Suppose you have 4 bills in your pocket, 1 is a ten dollar bill and 3 are five dollar bills. You will select two at random. Let $X$ be the total amount of money that you select.

(a) (10) Obtain the probability mass function (pmf) for $X$. For this problem, actually do the arithmetic, giving the probabilities as fractions. No calculator.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

(b) (5) Obtain the expected value of $X$, $E(X)$. Give the expected value as a fraction. No calculator.

$$E(X) = \frac{1}{2} \cdot 15 + \frac{1}{2} \cdot 10 = \frac{25}{2}$$

(c) (10) Now let $Y$ be the number of five dollar bills selected. Give the probability mass function for $Y$.

<table>
<thead>
<tr>
<th>$Y$</th>
<th>$P(Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
4. (20) You are driving to school. You take route A with probability 1/6 and route B with probability 5/6. If you take route A, you will encounter 3 stoplights and if you take route B you will encounter only two stoplights. There is probability 1/4 of hitting each of the stoplights on route A, independently, and probability 1/3 of hitting stoplights on route B, also independently. No arithmetic.

(a) (10) Calculate the probability of encountering no stop lights on the way to school. Justify steps.

\[
\Pr(\text{No lights}) = \Pr(\text{NL}(A)P(A) + \Pr(\text{NL}(B)P(B))
\]

\[
= \frac{1}{6} \cdot \left(\frac{3}{4}\right)^3 + \frac{5}{6} \cdot \left(\frac{2}{3}\right)^2
\]

(b) (10) What is the probability of hitting exactly one stop light? Justify steps.

\[
\Pr(\text{ex one}) = \Pr(\text{ex one}(A)P(A) + \Pr(\text{ex one}(B)P(B))
\]

\[
= \frac{1}{6} \cdot \binom{3}{1} \left(\frac{3}{4}\right)^2 + \frac{5}{6} \cdot \binom{2}{1} \cdot \frac{2}{3} \cdot \frac{1}{3}
\]

Because \# lights hit \(A \sim \text{Bin}(3, \frac{3}{4})\)

and \# lights hit \(B \sim \text{Bin}(2, \frac{1}{3})\)

5. (10) Ten individuals, five men and five women, will take a trip together and they will drive a total of 5 cars, with two individuals per car. Find the probability that only one car has exactly two men and only one car has exactly two women. Explain.

We need to put the men and women into 5 cars so that ex 2 in each.

\[
\binom{10}{2} \cdot \binom{8}{2} \cdot \binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2} = \binom{10}{2} \cdot \text{ways}
\]

\[
\Pr = \frac{5}{6} \cdot \binom{5}{2} \cdot \binom{3}{2} \cdot \binom{3}{2} \cdot \binom{2}{1} \cdot \binom{3}{2} \cdot \binom{1}{1} = \binom{10}{2} \cdot \text{ways}
\]

Put 3 men in 3 cars

Put 3 women in 3 cars
Theory Problems

6. (5) Prove that $\Pr(A_1 \cup A_2) \leq \Pr(A_1) + \Pr(A_2)$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \leq \Pr(A) + \Pr(B)$$

7. (5) Prove that $A \cap B = \emptyset$ implies that $A$ and $B$ cannot be independent.

Suppose $A \cap B = \emptyset$, then

$$\Pr(A \cap B) = \Pr(A) \Pr(B) > 0$$

But $A \cap B = \emptyset \Rightarrow \Pr(A \cap B) = 0$. So $A \cap B$ cannot be independent.

8. (5) Show that, if the events $A$ and $B$ are independent, then $A$ and $B^c$ are also independent. Hint: Use the Law of Total Probability

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B^c)$$

$$= \Pr(A) \Pr(B) + \Pr(A \cap B^c)$$

$$\Rightarrow \Pr(A \cap B^c) = \Pr(A) \Pr(B^c)$$

9. (10) Show that the variance of a binomial($n, p$) random variable is $\sigma^2 = np(1 - p)$. You may assume that you have already derived the fact that the mean is $\mu = np$. Justify.

Done in class + in book.