Math 227A take-home final exam

Due Fri, Dec 2, 5:00 pm, office Rowland Hall 540J

1. Consider the following nonlinear system (two-species competition):

\[
\begin{align*}
\dot{y}_1 &= r_1 y_1 \left( 1 - \frac{y_1 + \alpha_{12} y_2}{K} \right), \\
\dot{y}_2 &= r_2 y_2 \left( 1 - \frac{y_2 + \alpha_{21} y_1}{K} \right),
\end{align*}
\]

where \( r_i > 0 \) are the linear growth rates of the two species, \( K \) is the carrying capacity, and \( \alpha_{ij} > 0 \) describe how much each species influences the other species' growth (for \( i, j \in \{1, 2\} \)). (a) Find critical points and investigate their stability. (b) Fix the parameters \( r_1, r_2 \) and \( K \) and describe the bifurcations as you vary \( \alpha_{12} \) and \( \alpha_{21} \) one at a time. Draw the bifurcation diagrams. (c) Solve the ODEs numerically by using \( y_1(0) = 2, \ y_2(0) = 3, \ K = 200, \ r_1 = 1, \ r_2 = 1.5, \) and by picking the values for the parameters \( \alpha_{12} \) and \( \alpha_{21} \) before and after the bifurcation. Will the long-term behavior of the system be the same if you vary the initial conditions? Explain.

2. Consider the boundary value problem

\[ u'' - u = e^x, \quad u(0) = 1, \quad u'(1) = 0. \]

(a) Solve the problem by finding the Green’s function. What is \( u'(0) \)?
(b) Set up the method of shooting: (i) Rewrite the problem as an initial value problem with \( u'(0) = c \). (ii) Solve the IVP numerically by using Mathematica or other program for several values of \( c \). For each particular value of \( c \), evaluate \( F(c) \equiv u'(1) \). (iii) Plot the \( F(c) \) as a function of \( c \) to show that equation \( F(c) = 0 \) has a root, \( F(c_*) = 0 \). (iv) Show that \( c_* \) (approximately) coincides with \( u'(0) \) found in (a).