6.3. Show that
\[ \frac{\partial C_{\text{opt}}(\bar{P})}{\partial \bar{P}} = \lambda(\bar{P}) \]

Hint: First use Equation 6.11 to obtain \( \frac{1}{\lambda(\bar{P})} \), and then differentiate the right hand side of Equation 6.13 and substitute.

Solution: From (6.11) we observe that
\[ \frac{1}{\lambda(\bar{P})} \sum_{\{i \in \mathcal{H}, h_i > \sigma^2 \lambda(\bar{P})\}} g_i - \sum_{\{i \in \mathcal{H}, h_i > \sigma^2 \lambda(\bar{P})\}} g_i \frac{\sigma^2}{h_i} = \bar{P} \]

With reference to Figure 6.3, let \( \bar{P} \) be such that \( \frac{1}{\lambda(\bar{P})} \in \left(\frac{1}{h_i}, \frac{1}{\bar{h} + 1}\right) \), for some \( j, 1 \leq j \leq J \), with \( \frac{\sigma^2}{h_i - \bar{h}} := \infty \). In this interval, we see from the above equation that
\[ \frac{\partial \frac{1}{\lambda(\bar{P})}}{\partial \bar{P}} = \left( \sum_{\{i \in \mathcal{H}, \frac{\sigma^2}{h_i} \leq \frac{\sigma^2}{\bar{h}}\}} g_i \right)^{-1} \]

Now writing (6.13) for the same value of \( \bar{P} \), we have
\[ C_{\text{opt}}(\bar{P}) = \kappa \sum_{\{i \in \mathcal{H}, \frac{\sigma^2}{h_i} \leq \frac{\sigma^2}{\bar{h}}\}} g_i \ln \left( \frac{h_i}{\sigma^2 \lambda(\bar{P})} \right) \]

Differentiating, with respect to \( \bar{P} \), we obtain
\[ \frac{\partial C_{\text{opt}}(\bar{P})}{\bar{P}} = \kappa \sum_{\{i \in \mathcal{H}, \frac{\sigma^2}{h_i} \leq \frac{\sigma^2}{\bar{h}}\}} g_i \frac{\sigma^2 \lambda(\bar{P})}{h_i} \frac{h_i}{\sigma^2} \frac{\partial \frac{1}{\lambda(\bar{P})}}{\partial \bar{P}} \]

i.e.,
\[ \frac{\partial C_{\text{opt}}(\bar{P})}{\bar{P}} = \kappa \lambda(\bar{P}) \frac{\partial \frac{1}{\lambda(\bar{P})}}{\partial \bar{P}} \sum_{\{i \in \mathcal{H}, \frac{\sigma^2}{h_i} \leq \frac{\sigma^2}{\bar{h}}\}} g_i \]
\[ = \kappa \lambda(\bar{P}) \]

where we have used the expression for \( \frac{\partial \frac{1}{\lambda(\bar{P})}}{\partial \bar{P}} \) derived earlier.
6.4. Solve the optimization problem (6.6, 6.7) using the KKT Theorem (Theorem C.3 in Appendix C) and obtain all the above results.

**Solution:** We have a problem of maximizing a strictly concave function subject to linear constraints. Hence, it is necessary and sufficient that the optimum solution is a KKT point. This can easily be checked for the solution obtained.

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**Problems**

6.1. For the single-user, single-carrier problem, described in the text, consider the problem

\[
\min \sum_{i_j \in N} P_j \ g_j
\]

subject to

\[
\sum_{i_j \in N} \gamma \ln \left( 1 + \frac{h_j P_j}{\sigma^2} \right) g_j \geq a
\]

where \(a > 0\).

a. Characterize the optimal solution of this problem.

b. Let the optimum value of this problem be denoted by \(P_a\). Show that \(\tilde{P}_a = P_a\) (defined in the text).

**Solution:** Define
a. 

\[ L(P, \eta) = \sum_{h_j \in \mathcal{H}} P_j g_i - \eta \sum_{h_j \in \mathcal{H}} \left( \ln \left( 1 + \frac{h_j P_j}{\sigma^2} \right) \right) g_j \]

where \( \eta \) is nonnegative. Rearranging

\[ L(P, \eta) = \sum_{h_j \in \mathcal{H}} \left( P_j - \eta \ln \left( 1 + \frac{h_j P_j}{\sigma^2} \right) \right) g_j \]

This is of exactly the same form as in the problem of rate maximization subject to a power constraint. We use positive power in state \( h_j \) provided

\[ \frac{\sigma^2}{h_j} < \eta \]

where \( \eta \) may be interpreted as revenue per unit rate. When \( \frac{\sigma^2}{h_j} < \eta \), the optimal power allocation is given by \( (\eta - \frac{\sigma^2}{h_j})^+ \); i.e., the optimal power allocation is given by

\[ P_{\eta,j} = \left( \eta - \frac{\sigma^2}{h_j} \right)^+ \]

Let

\[ \bar{P}_\eta = \sum_{h_j \in \mathcal{H}_j} \left( \eta - \frac{\sigma^2}{h_j} \right)^+ g_j \]

Let the rate thus obtained be \( C_\eta \) which is seen to be

\[ C_\eta = \sum_{\{h_j \in \mathcal{H}, \eta > \frac{\sigma^2}{h_j}\}} \gamma \left( \ln \left( \frac{h_j \eta}{\sigma^2} \right) \right) g_j \]

Notice that for increasing \( \eta \) \( C_\eta \) increases, with a corresponding increase in \( \bar{P}_\eta \). Thus to solve this problem \( \eta \) is chosen to be \( \eta(a) \) give by

\[ C_{\eta(a)} = a \]

and

\[ \bar{P}_a = \bar{P}_{\eta(a)} \]

b. To be done