7.3. Consider the Aloha protocol for multiple access networks. Let $X$ be the number of nodes that are backlogged at the beginning of a slot. Assume $X$ has a Poisson distribution with mean $\lambda$. Now assume that each backlogged node transmits in the slot with probability $1/\lambda$ independent of the others.

a. Obtain the joint probability of $k$ nodes being backlogged and $r$ transmitting. Also obtain the unconditional probability that the slot is idle.

b. If the slot was observed to be idle, what is the \textit{a posteriori} probability that $k$ nodes were backlogged at the beginning of the slot?

c. Similarly, find the \textit{a posteriori} probability that given that there was a successful transmission in the slot, there were $k$ backlogged nodes at the beginning of the slot.

d. Using these results suggest a method to continuously estimate $\lambda$ based on the event in a slot—success or idle. Suggest an estimation method for $\lambda$ when a collision is observed in a slot.

Solution:

a. The conditional probability of $r$ attempting to transmit given $k$ are backlogged is given by

$$\binom{k}{r} \left( \frac{1}{\lambda} \right)^r \left( 1 - \frac{1}{\lambda} \right)^{k-r}$$

and the joint probability of $k$ and $r$ is

$$\binom{k}{r} \left( \frac{1}{\lambda} \right)^r \left( 1 - \frac{1}{\lambda} \right)^{k-r} e^{-\lambda} \frac{\lambda^k}{k!}$$

for $r \geq k$ and $k > 0$. The slot is idle if none of the given backlogged nodes transmit, i.e.,

$$\sum_{k=0}^{\infty} \left( 1 - \frac{1}{\lambda} \right)^k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} (\lambda - 1)^k \frac{1}{k!}$$

$$\quad = e^{-\lambda} e^{\lambda - 1} = e^{-1}$$
b. 
\[
\Pr(k \text{ bklog } | \text{ idle}) = \frac{\Pr(\text{idle} | k \text{ bklog}) \times \Pr(k \text{ bklog})}{\Pr(\text{idle})}
\]
\[
= \frac{(1 - \frac{1}{\hat{x}})^k e^{-\hat{x}\hat{x}^k/k!}}{e^{-1}}
\]
\[
= \frac{\hat{x} - 1}{k!} e^{-(\hat{x} - 1)}
\]
i.e., the \textit{a posteriori} probability of there being \(k\) backlogged nodes after an idle slot on the network has a Poisson distribution with mean \((\hat{x} - 1)\).

c. The probability that the slot is successful is given
\[
\sum_{k=0}^{\infty} k \left(\frac{1}{\hat{x}}\right) \left(1 - \frac{1}{\hat{x}}\right)^{k-1} e^{-\hat{x}\hat{x}^k/k!} = e^{-\hat{x}} \sum_{k=1}^{\infty} \frac{(\hat{x} - 1)^{k-1}}{(k-1)!}
\]
\[
= e^{-\hat{x}} e^{\hat{x} - 1} = e^{-1}
\]
Using exactly the same method as above, we can obtain the \textit{a posteriori} probability of there being \(k\) backlogged nodes following a success (or \(k + 1\) backlogged nodes at the beginning of a slot in which there was a success) as under.
\[
\Pr(k + 1 \text{ bklog } | \text{ suxes}) = \frac{\Pr(\text{suxes} | k + 1 \text{ bklog}) \times \Pr(k + 1 \text{ bklog})}{\Pr(\text{suxes})}
\]
\[
= \frac{(k + 1) \left(\frac{1}{\hat{x}}\right) \left(1 - \frac{1}{\hat{x}}\right)^k e^{-\hat{x}\hat{x}^{k+1}/(k+1)!}}{e^{-1}}
\]
\[
= \frac{(\hat{x} - 1)^k}{k!} e^{-(\hat{x} - 1)}
\]
i.e., in the slot \textit{following} a successful transmission, the number of backlogged nodes has a Poisson distribution with mean \((\hat{x} - 1)\).

d. Clearly, we should decrease the estimate of the number of backlogged nodes by 1 following a successful or an idle slot. However since there are fresh arrivals that can occur in the slot the update of the estimate for the number of backlogged nodes in slot \((n + 1)\), \(\hat{x}_{n+1}\), should be
\[
\hat{x}_{n+1} = \max\{\lambda\hat{x}_n + -1\lambda\}\]
following a success or an idle. The addition of $\lambda$ accounts for the new arrivals. The situation following a collision is not so straightforward. For this case it can be shown the estimate update should
\[
\hat{x}_{n+1} = \hat{x}_n + \lambda + \frac{1}{2 - e}
\]

7.6. Consider a multiple access channel in which the following situation arises. Two nodes $A$ and $B$ are ready to send a packet at the same time. This typically happens immediately following a successful transmission. In the $k$-th round after $(k-1)$ collisions have occurred, the nodes wait for a random period of $w \in [0, 1, \cdots, 2^{k-1} - 1]$ slots of time, with each of the $2^{k-1}$ choices being equally likely. Let $c_k$ be the probability of a collision in round $k$ given that the previous $(k-1)$ rounds had a collision.

a. Find $c_k$ as a function of $k$ for all $k$.

b. What is the probability that round $k$ lasts $n$ slots? What is its mean? Assume that a collision has occurred in the round.

c. Find the probability that round $k$ lasts $n$ slots. Do not assume that collision occurred. Also find the mean number of slots in round $k$.

d. Find $p_k$, the probability that exactly $k$ rounds are needed to resolve a collision involving only two nodes and no new nodes transmitting until the collision is resolved.

e. Assume that the collision is resolved in favor of $A$ in the third round. In this case $A$ will reset its collision counter. Assume that the packet being transmitted by $A$ is longer than the backoff time chosen by $B$. Because of 1-persistence, $B$ will transmit soon after $A$'s transmission. Now if $A$ has another packet to transmit, there will be a collision immediately following $A$'s successful transmission and both $A$ and $B$ will increase their collision counters. What is the probability that this collision is resolved in favor of $A$ in the second round?
Solution:

a. There is a collision in round $k$ if both nodes pick the same integer from $[0, 2^{k-1} - 1]$. Thus this probability is $(1/2^{k-1})^2$. $2^{k-1} = 1/2^{k-1}$.

b. Given that a collision occurred, both will have picked the same number in the said interval. The probability of the round lasting $n$ slots is $1/2^{k-1}$ for $n = 1, \ldots, 2^{k-1}$. The mean is $(1/2^{k-1}) \cdot (2^{k-1} / 2) = 2^{k-2} + 1/2$.

c. The duration of the round is a random variable that is the minimum of two uniformly distributed numbers. Thus,
$$P(X=n, Y=n) + P(X=n, Y>n) + P(Y=n, X>n) = (1/2^{k-1})^2 + 2 \cdot (1/2^{k-1}) \cdot (2^{k-1} / 2) = (2/2^{k-1}) [1 - (n-1/2)/2^{k-1}]$$. The mean is $\frac{2}{2^{k-1}} \sum_{n=1}^{2^{k-1}} n \left[1 - \frac{n-1/2}{2^{k-1}}\right]$.

d. 
$$p_k = (1 - c_k) \prod_{i=1}^{k-1} c_j$$

e. The collision counter of $A$ will be one while that of $B$ will be four. $A$ will lose if it picks 1 and $B$ picks 0. This happens with probability $1/2 \cdot 1/16$. The probability of a collision is $2 \times \frac{1}{2} \cdot \frac{1}{16}$ corresponding to both picking 0 or both picking 1. In all other cases $A$ will win. Hence this probability is $\frac{28}{32}$. 

\[\]