Announcements

- Skip Lists
Skip Lists

• Problem: searching a linked list takes tool long $O(n)$
• Want to search more quickly
• Solution: add more edges that let us skip through items
Skip List

$S_2$: -inf $\rightarrow$ 5 $\rightarrow$ +inf

$S_1$: -inf $\rightarrow$ 5 $\rightarrow$ 9 $\rightarrow$ +inf

$S_0$: -inf $\rightarrow$ 2 $\rightarrow$ 5 $\rightarrow$ 7 $\rightarrow$ 9 $\rightarrow$ +inf
Skip List

- Consists of a series of lists \( \{S_0, S_1, \ldots, S_h\} \)
- \( S_0 \) contains every item
- For \( i = 1, \ldots, h-1 \) \( S_i \) contains a randomly selected subset of \( S_{i-1} \) plus \(-\infty\) and \(+\infty\)
- \( S_h \) contains only \(-\infty\) and \(+\infty\)
Positions

after(p) - position after p on the same level
before(p) - position before p on the same level
below(p) - position below p on the same tower
above(p) - position above p on the same tower
Searching for $k$ in a Skip List

Finds the largest key $\leq k$

Start at top-most level in the left most position $p$
while $\text{below}(p)\neq\text{null}$ do
    $p\leftarrow\text{below}(p)$  //drop down
    while $\text{key}(\text{after}(p))\leq k$ do
        $p\leftarrow\text{after}(p)$  //scan forward
    return $p$
Skip List

Search for 7

$S_0$: 
- inf $\rightarrow$ 2 $\rightarrow$ 5 $\rightarrow$ 7 $\rightarrow$ 9 $\rightarrow$ +inf

$S_1$: 
- inf $\rightarrow$ 5 $\rightarrow$ 9 $\rightarrow$ +inf

$S_2$: 
- inf $\rightarrow$ 5 $\rightarrow$ +inf
Skip List

Search for 7

\[ S_0 \]

- inf  \quad 2  \quad 5  \quad 7  \quad 9  \quad +inf

\[ S_1 \]

- inf  \quad 5  \quad 9  \quad +inf

\[ S_2 \]

- inf  \quad 5  \quad +inf
Skip List

Search for 7
Skip List

Search for 7

$S_2$: \(-\infty\) - 5 - \(+\infty\)

$S_1$: \(-\infty\) - 5 - 9 - \(+\infty\)

$S_0$: \(-\infty\) - 2 - 5 - 7 - 9 - \(+\infty\)
Skip List

Search for 7

\[
\begin{align*}
S_2 & : -\infty \rightarrow 5 \rightarrow +\infty \\
S_1 & : -\infty \rightarrow 5 \rightarrow 9 \rightarrow +\infty \\
S_0 & : -\infty \rightarrow 2 \rightarrow 5 \rightarrow 7 \rightarrow 9 \rightarrow +\infty
\end{align*}
\]
Skip List

Search for 7

$S_2$: 
- $\text{inf}$ --> 5 --> +$\text{inf}$

$S_1$: 
- $\text{inf}$ --> 5 --> 9 --> +$\text{inf}$

$S_0$: 
- $\text{inf}$ --> 2 --> 5 --> 7 --> 9 --> +$\text{inf}$
Insertion of k

p=Search for k using search procedure
Add k after item p at bottom level
while random()<1/2 do
    while above(p)=null do
        p<-before(p)
    p<-above(p)
    insert after item p at next higher level
Removal of $k$

1. Find $k$
2. Remove $k$ from bottom level
3. Look at next level up
4. If $k$ is present in this level, remove $k$ from this level otherwise exit
5. Goto 3
Cost

• Expected height

Each level has half the expected number of entries as the previous one

\[ P_i \leq n/2^i \]

⇒ Expected number of levels is \( O(\log(N)) \)

Book has more formal reasoning
Search Time

- Outer loop executes $O(h)$ which with high likelihood is $O(\log n)$
- Likely to make $O(1)$ operations on given level
- Only considering keys on level $i$ between the current key and the next greater one on level $i+1$
- Half of these keys in the range $[k, \text{next}(k \text{ on level } i+1)]$ should appear in level $i+1$ and only $k$ and $\text{next}(k \text{ on level } i+1)$
- Expect to scan small constant number of keys on each level $= O(1)$
- Total search $O(\log n)$
Space Usage

- Bottom level \( n \)
- Next level \( n/2 \)
- Next level \( n/2^2 \)
- Sum of \( n+n/2+n/4...=n(1+1/2+1/4...)=2n \)
- Space=\( O(n) \)