Engineering Analysis

MAE 200A

Supplement: Examples Problems and Solutions

I. General Ordinary Differential Equations – Basics and Numerical Solution Methods
II. Linear Algebra
III. Linear Time Invariant (LTI) ODEs
IV. Linear Time Varying (LTV) ODEs
V. Nonlinear Algebraic Equations
VI. Nonlinear ODEs

C1: Continuing Example: The Planar Circular Restricted Three-Body Problem
C2: Continuing Example: ODE with Parameter Bifurcation

© 2011 by authors E. Trumbauer and K.D. Mease
Note to the Reader

This material is meant to supplement the lectures and homework in response to the common request from students to see more examples. Example problems are given along with solutions. The examples provide greater diversity and level of complexity than what can reasonably be covered in a lecture.
I. GENERAL ORDINARY DIFFERENTIAL EQUATIONS –
BASICS AND NUMERICAL METHODS

Separation of Variables, Uniqueness

Consider the system defined by \( \dot{x} = x^{3/5}, x(0)=0, t \geq 0 \).

Part 1: This IVP can be solved by hand. Find a nontrivial solution to the problem. (Hint: Use separation of variables)

Part 2: Use the some of the same integrators from the first homework assignment (Euler, Modified Euler, Runge-Kutta, ode45) to solve this problem numerically. How does this compare to the analytical solution? What happens to the numerical solutions if \( x(0)=0.0001 \)?

Part 3: Perform the first step of Euler's method by hand. What is causing the discrepancy we are seeing? What condition is not being met?

Solution:

Part 1: Using the standard separation of variables method and rearranging terms we get

\[
\frac{dx}{dt} = x^{3/5} \rightarrow x^{-3/5} \, dx = dt \rightarrow \frac{5}{2} x^{2/5} = t + C \rightarrow x^{2/5} = \frac{2}{5} \left( t + C \right) \rightarrow x = \left[ \frac{2}{5} \left( t + C \right) \right]^{5/2}
\]

Applying the initial condition \( x(0)=0 \) we must have that \( C=0 \) and so \( x(t) = \left[ \frac{2}{5} t \right]^{3/2} \) is a solution to the initial value problem.

Part 2:
All of the numerical methods presented so far will return a constant zero value for the IVP when \( x(0)=0 \), which does not track with our analytical solution at all. On the other hand, letting \( x(0)=0.0001 \) makes the numerical methods behave as expected, with the better integrators more closely tracking the analytical solution.

This is an easy system, so do try to make your own code for this, or adapt your previous work from the first assignment. However, here is some sample code:

```matlab
clear all;

T=5.0; % T is the final time, and also the integration time since the initial time is zero
x0=0.0001; %initial condition

%Analytical Solution
N=T*100 + 1;
for i=1:1:N
    time=(i-1)*0.01;
    tsteps(i)=time;
    xexact(i)=(2/5*(time+5/2*x0))^(5/2);
end
figure(1)
plot(tsteps,xexact,'k');
hold on;

%Euler Integration
% tE(1)=0;
% xE(1)=x0;
% dtE=0.25; % Be sure that T/dtE is an integer
imax=T/dtE + 1;
for i=2:1:imax
    tE(i)=(i-1)*dtE;
    xdot(i-1)=xE(i-1)^(3/5);
    xE(i)=xE(i-1)+dtE*xdot(i-1);
end
plot(tE,xE(1,:),'r');
hold on;

%Modified Euler Integration
% tME(1)=0;
% xME(1)=x0;
% dtME=0.5; % Be sure that T/dtME is an integer
imax=T/dtME + 1;
for i=2:1:imax
    tME(i)=(i-1)*dtME;
    x_pred=xME(i-1)+(xE(i-1)^(3/5))*dtME;
    xdot_approx=0.5*([xE(i-1)^(3/5)]+[x_pred^(3/5)]);
    xME(i)=xE(i-1)+dtME*xdot_approx;
end
plot(tME,xME(1,:),'c');
hold on;

% Runge-Kutta Integration
% tRK(1)=0;
% xRK(1)=x0;
% dtRK=0.5; % Be sure that T/dtRK is an integer
imax=T/dtRK + 1;
for i=2:1:imax
    tRK(i)=(i-1)*dtRK;
```
\begin{verbatim}

k1=f_Easy(tRK(i-1),xRK(i-1));
k2=f_Easy(tRK(i-1)+0.5*dtRK,xRK(i-1)+0.5*k1*dtRK);
k3=f_Easy(tRK(i-1)+0.5*dtRK,xRK(i-1)+0.5*k2*dtRK);
k4=f_Easy(tRK(i),xRK(i-1)+k3*dtRK);
xRK(i)=xRK(i-1)+dtRK*(k1+2*k2+2*k3+k4)/6;
end
plot(tRK,xRK(1,:),'b');
hold on;

% Matlab Integration Routine
% options = odeset('RelTol',1e-0,'AbsTol',1e-0);
[t,x45]=ode45('f_Easy',[0 T],x0,options);
plot(t,x45(:,1),'g');
hold off;

Additionally:
function xdot=f_Easy(t,x)
xdot=[x(1)^(3/5)];

Part 3:

For the first step of Euler's method, we have that \( x_1=x_0+h f(x_0)=0+h \times 0=0 \) for any stepsize \( h \). Is this a problem with Euler's method and the other numerical method? The issue is that the numerical methods are approximating \( x(t)=0, t \geq 0 \), which is a different solution for the IVP! By letting \( x(0)=0.0001 \) the possibility of two solutions is eliminated which is why things work as normal. The difference between the two cases is that the original IVP is not guaranteed to have unique solutions because the Lipschitz condition is not satisfied at \( t=0 \) (verify this). When presented with an unfamiliar system, it is sometimes worth it to examine the conditions of existence and uniqueness.
\end{verbatim}
The Lorenz System

The Lorenz system is a famous example of a system that is very sensitive to initial conditions and errors, a.k.a. chaotic. It is based on a simplified model of convection in the atmosphere. The system of equations for a particular case is:

\[
\begin{align*}
\dot{x} &= 10(y - x) \\
\dot{y} &= x(27 - z) - y \\
\dot{z} &= xy - 8/3 \, z
\end{align*}
\]

Use Matlab's built in solvers ode23, ode45, and ode113 to integrate this system for 30 units of time. Plot one or more of the coordinates as a function of time.

Solution and Discussion

```matlab
function Lorenz_System()

%******** Initial condition and timespan ********
time_span = [0 30];
x0 = [1.00 0.00 0.00]
[t_23, x_23] = ode23(@lorenz_equations, time_span, x0);
[t_45, x_45] = ode45(@lorenz_equations, time_span, x0);
[t_113, x_113] = ode113(@lorenz_equations, time_span, x0);

%************** Plots **************

% Plot x versus time
figure(1)
hold on
title('Lorenz System: ODE23 (Green), ODE45 (Blue), ODE113 (Red)')
ylabel('x coordinate')
xlabel('t')
plot(t_23,x_23(:,1),'g')
plot(t_45,x_45(:,1),'b')
plot(t_113,x_113(:,1),'r')
hold off

% Plot z versus x, the famous butterfly image
figure(2)
hold on
title('Lorenz System: ODE23 (Green), ODE45 (Blue), ODE113 (Red)')
ylabel('z coordinate')
xlabel('x coordinate')
plot(x_23(:,1),x_23(:,3),'g')
plot(x_45(:,1),x_45(:,3),'b')
plot(x_113(:,1),x_113(:,3),'r')
hold off
end

function dy = lorenz_equations(t,y)
dy = zeros(3,1);
```
\begin{verbatim}
X = y(1);
Y = y(2);
Z = y(3);
dy(1) = 10*(Y-X);
dy(2) = X*(27-Z) - Y;
dy(3) = X*Y - 8/3*Z;
end
\end{verbatim}

Numerical integrators have finite step lengths, so any error will lead to a difference between the starting value at the next step and the true solution. For simple or linear systems, using Matlab's built in integrators and default settings is frequently sufficient and much can be learned that way. However, in more complicated systems, more thought should be paid to which integration methods, accuracy, and the structure of the system. In some problems, one can see the benefit of adjusting the error tolerances until a degree of accuracy appropriate to the problem is reached. An example of this is in the Continuing Example: Planar Restricted Three Body Problem later in this section. However, in some systems no matter the accuracy there will be large differences between numerical results in the long term regardless of the method used, though over some relevant time span the results may be useful. The Lorenz system is an example of a chaotic system where this is the case.

Throughout this class, the study of features such as equilibrium points, periodic orbits, stability, regions of attraction, etc. give a qualitative understanding of systems that is needed to complement the quantitative numerical simulations. Other systems where this qualitative viewpoint is especially needed in light of numerical difficulties: multibody dynamics in spacecraft, turbulent flows, cardiac cycles, economics, population growth, earthquakes, and even the lowly double pendulum. In the image below, even though as expected the trajectories differ,
by taking a different viewpoint than time plots structure emerges that can tell us about the nature of solutions. This viewpoint will be developed throughout the class.
**Continuing Example: Circular Restricted Three-Body Problem - Equations of Motion**

This system will be used as a source of examples throughout the class. Read the Introduction and do Problem 1 in the Continuing Examples section. Also, once the system is in first order form, try to make a program to integrate a couple trajectories.

**Continuing Example: Circular Restricted Three-Body Problem – Error Tolerances**

Read the Introduction and do Problem 2 in the Continuing Examples section. This problem discusses changing error tolerances in Matlab integration.

**Continuing Example: ODE with Parameter Bifurcation - Classification and Equilibria**

This is another system that will be used later in the class. Do Problem 1 of the example in the Continuing Examples section.
II. LINEAR ALGEBRA

Other uses for matrices: Power Grid Analysis

Matrices can represent much more than just transformations in physical space. Aside from clean and efficient energy production, studying the structure of a power grid (connections where energy may be transmitted from place to place) is important to reduce energy loss. Matrices can represent these connections. Drawing the \( n \) locations and their connections as a graph, the \( ij \)-th entry of the \( n \times n \) Adjacency Matrix is a 1 if there is a direct connection between two points, and 0 otherwise.

SYSTEM 1 – Just to get the idea

Part 1: For the following graph, create the adjacency matrix \( A \).

![Graph](Image)

Part 2: Calculate \( A^2 \). Create a new matrix \( B \) where the entries \( b_{ij} \) are equal to the numbers of paths of exactly length 2 (2 segments) connecting Node \( i \) with Node \( j \). What is the relation between \( A^2 \) and \( B \)? Explain why this is the case, mentioning the formula for matrix multiplication and what it means in this context?

SYSTEM 2

Part 1: For the following graph, create the adjacency matrix \( A \).
**Part 2:** Calculate $A^2, A^3$ in Matlab. What does it mean for the $ij$-th entry of $A^k$ to be nonzero? What does the particular value mean? Verify for some nonzero entries that this matches up with the actual result. If node $i$ is a power plant and node $j$ an important energy consumer, would a higher or lower value of the $ij$-th entry of $A^k$ be desirable? Why?

**Part 3:** An invariant subspace of a matrix $A$ is a subspace $W$ of the vector space $V$ such that $Aw \in W$ for all $w \in W$. A simple example is the line spanned by an eigenvector. (As an aside, how many independent eigenvectors does $A$ have? Do not use Matlab for this.) Looking at the graph, can you guess any higher dimensional subspaces? In practical terms of energy transmission, what does it mean if two nodes are in different subspaces?

**Solution**

**System 1**

**Part 1**

Node 1 only has a direct connection to Node 2 and not to itself or Node 3. This means that $a_{11}=0, a_{12}=1, a_{13}=0$. Node 2 connects with Node 1 and Node 3 but not with itself, so we have $a_{21}=1, a_{22}=0, a_{23}=1$. Node 3 connects with Node 2 but not with itself or Node 1, so we have $a_{31}=0, a_{32}=1, a_{33}=0$. This gives the adjacency matrix as:

$$
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 
\end{bmatrix}
$$

**Part 2**

$A^2 =$

$$
\begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
1 & 0 & 1 
\end{bmatrix}
$$

Let's calculate $B$. From Node 1, a path $1 \rightarrow 2 \rightarrow 1$ exists and is the only way to get from Node 1 back to itself in 2 steps. Thus $B_{11}=1$. While Node 1 and Node 2 are adjacent and can be connected in one step, there is no way to connect them with exactly two steps, so $B_{12}=0$. From Node 1 to Node 3 we have a single path $1 \rightarrow 2 \rightarrow 3$ so $B_{13}=0$. 
From Node 2 to itself there are two paths $2 \rightarrow 1 \rightarrow 2$ and $2 \rightarrow 3 \rightarrow 2$ so $B_{22} = 2$.

From Node 2 to Node 3 there is a path of length 1, but none of length 2, so $B_{23} = 2$.

From Node 3 to itself there is just one path $3 \rightarrow 2 \rightarrow 3$ so $B_{33} = 1$.

Since we just care about nodes being connected and are not using directed graphs where flows only go one way, it should be clear that these matrices must be symmetric. Thus the information above is sufficient to see that $A^2 = B$. This should not be too surprising, after all if two nodes $i$ and $j$ are connected by a path of exactly length 2, then there must be two paths of length 1 where the end of one connection is the beginning of the other and the endpoints are $i$ and $j$. For $A^2$ the $ij$-th entry is equal to $\sum_1^3 a_{ik} a_{kj}$. Let's consider $A_{22}^2 = \sum_1^3 a_{2k} a_{k2}$.

$a_{21} a_{12} = 1 \times 1 = 1$ because connections exist from Node 1 to 2, and obviously then from 2 to 1. This hold also for $a_{23} a_{32} = 1 \times 1 = 1$. On the other hand, $a_{22} a_{22} = 0 \times 0 = 0$. Thus $A_{22}^2 = \sum_1^3 a_{2k} a_{k2} = 1 + 0 + 1 = 2$.

**System 2**

**Part 1:** By inspection following the process System 1 of looking for direct links, we have:

$$A = 
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}$$

**Part 2:**

$$A^2 = 
\begin{bmatrix}
2 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 2 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 3 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$
Looking at the matrices for $A$, $A^2$, $A^3$ we can see a nonzero entry in $A^k$ means there is at least one path with $k$ links – a path of length $k$ – that connects node $i$ with node $j$. The specific number is the number of paths with length $k$ connecting the two nodes. This can be seen more systematically by looking at the definition of matrix multiplication. For example, for $A^2$ the $ij$-th entry is equal to $\sum_l a_{il} a_{lj}$ which is equal to the number of paths from node $i$ to $j$ that pass through some node $l$ first.

Thinking about energy transmission, a higher value in an entry corresponds to more available routes to transmit electricity, i.e. creates some redundancy. This is useful not only to avoid overloading a connection, but also in case of natural disasters where one or more connections may be destroyed.

**Part 3:** Looking at the graph, it should be clear that after a couple iterations, a connection will exist between nodes 1,2,4, and 5. The same holds between nodes 3, 6, and 7. However, since these two groups are not connected, no connection exists between them. This can be stated in terms of invariance saying that

$$W_1 = \text{span} \{(1,0,0,0,0,0,0,0),(0,1,0,0,0,0,0,0),(0,0,0,1,0,0,0,0),(0,0,0,0,1,0,0,0)\}$$

$$W_2 = \text{span} \{(0,0,1,0,0,0,0,0),(0,0,0,0,0,1,0,0),(0,0,0,0,0,0,0,1)\}$$

are two invariant subspaces of $A$. Invariant subspaces then correspond to the existence of distinct connected elements, i.e. no power may be transmitted between them.
Least Squares Example 1: Average Air Density at a Wind Turbine

On a series of clear days measurements were taken of the power generated by a wind turbine. This highly efficient turbine generates power given by \( P = \frac{8}{27} \rho S v^3 \) where \( \rho, v \) are the air density (in kg/m\(^3\)) and velocity at the blades (m/s), respectively, and \( S \) is the swept rotor area (in sq. m). The turbine has blades of length 10 m. The measured velocities were 5 m/s, 7 m/s, and 4 m/s. The corresponding power measurements in watts are 14,000, 38,000, and 7,100. What is the least squares estimate of the air density on a clear day at the site of this turbine?

Solution

The only unknown value is the air density, which appears linearly in the formula for power. We have the following linear equations in terms of density

\[
\frac{8}{27} \pi (10 \text{ m})^2 (5 \text{ m/s})^3 \rho = 14000 \text{ W}
\]

\[
\frac{8}{27} \pi (10 \text{ m})^2 (7 \text{ m/s})^3 \rho = 38000 \text{ W}
\]

\[
\frac{8}{27} \pi (10 \text{ m})^2 (4 \text{ m/s})^3 \rho = 7100 \text{ W}
\]

which are inconsistent so there is no exact solution for all three, which is to be expected since air density is not generally constant and because there will always be some measurement error in real life.

Writing this in standard matrix form we have

\[
\begin{bmatrix}
\frac{8}{27} \pi 10^2 & \frac{5^3}{7^3} \\
\frac{8}{27} \pi 10^2 & \frac{7^3}{4^3} \\
\frac{8}{27} \pi 10^2 & \frac{4^3}{3^3}
\end{bmatrix} \begin{bmatrix} \rho \end{bmatrix} = \begin{bmatrix} 14000 \\
38000 \\
7100
\end{bmatrix}
\]

where \( A = \frac{8}{27} \pi 10^2 \begin{bmatrix} 5^3 \\
7^3 \\
4^3
\end{bmatrix} \), \( x = [\rho] \), \( b = \begin{bmatrix} 14000 \\
38000 \\
7100
\end{bmatrix} \). Keeping track of matrix dimensions we see that in this case \( (A^T A)^{-1} A^T b \) is a scalar (please verify this dimension counting by hand). The result is our least squares best estimate of the average density of \( \rho \approx 1.1917 \text{ kg/m}^3 \), slightly less than the 'standard' air density at sea level.
Least Squares Example 2: Curve Fitting for a Non-Newtonian Fluid

You have been asked to determine the approximate relationship between velocity and shear stress in a non-Newtonian fluid (meaning that the relationship is known to be nonlinear, i.e. $\tau \neq c\nu$ for some constant $c$). Measurements were taken at velocities [0.5 1.0 1.5 2.0 2.5]. The estimated shear stresses at these points are [.75 1.6 2.3 2.32 3.1]. The shear stress is assumed to be a function of $\nu$.

**Part 1:** Approximate the data with a fourth degree polynomial using the Least Squares Method in MATLAB. Plot the polynomial approximation on the interval 0 to 3 and compare to the data points. What is the maximum difference between the estimated and measured values at these points? Does this make this polynomial a good fit?

**Part 2:** Next, approximate the data with a second degree polynomial. Compare to the data points, what is the maximum difference between the estimated and measured values? How does this compare to the fourth order approximation? Is this necessarily worse? Would knowing the measurement accuracy help us decide which to use? What additional measurements would be helpful?

**Solution**

A common use for least squares approximations is approximating functions with polynomials. The most frequently seen example of this is linear regression. Although higher degree polynomials are not linear in their input variable, for a specific input value we can look at a polynomial as a linear combination of its coefficients. For example, at $\nu=2$ above, a fourth degree polynomial can be written as $c_4 2^4 + c_3 2^3 + c_2 2^2 + c_1 2 + c_0 = 16 c_4 + 8 c_3 + 4 c_2 + 2 c_1 + c_0$. Repeating this for each data point gives us a set of equations linear in the coefficients. Generally there are more data points than the degree of the polynomial you are trying to fit with, so this can be set up as a least squares problem.

(continued next page)
Writing the above example in the standard $Ax=b$ form we have:

Fourth degree approximation setup

$$\begin{bmatrix} 1 & .5 & .5^2 & .5^3 & .5^4 \\ 1 & 1 & 1^2 & 1^3 & 1^4 \\ 1 & 1.5 & 1.5^2 & 1.5^3 & 1.5^4 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 2.5 & 2.5^2 & 2.5^3 & 2.5^4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} .75 \\ 1.6 \\ 2.3 \\ 2.32 \\ 3.1 \end{bmatrix}$$

Second degree approximation setup

$$\begin{bmatrix} 1 & .5 & .5^2 \\ 1 & 1 & 1^2 \\ 1 & 1.5 & 1.5^2 \\ 1 & 2 & 2^2 \\ 1 & 2.5 & 2.5^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} .75 \\ 1.6 \\ 2.3 \\ 2.32 \\ 3.1 \end{bmatrix}$$

We can then use the least squares method to find the coefficients that minimize the squared error. This results in a fourth order approximation $\tau \approx 1.3133 \nu^4 - 7.2733 \nu^3 + 13.3117 \nu^2 - 8.0017 \nu + 2.2500$ and a quadratic approximation $\tau \approx -.2343 \nu^2 + 1.7869 \nu - .0220$.

Plotting the fourth order approximation we get

![Plot of Fourth Degree Polynomial Approximation](image.png)

In this case, the polynomial exactly equals the measurements at every point. This isn't surprising since we had a square, invertible matrix in the fourth order case.

But is this a realistic fit for the data? It does not seem realistic that the shear stress would spike upwards as the velocity goes to zero. The general fluctuations look a bit suspicious as well. Let's compare to the second degree approximation.
In this case we see that the polynomial does not exactly fit every measurement, and so the error at each data point is greater than in the fourth degree case. However, this looks like a better estimation for the relationship between velocity and shear stress. There is no increase in stress as the velocity goes to zero (having no stress as one approaches zero makes intuitive sense as well, though as seen in “Simple Bingham” fluids this is not strictly necessary), and fewer fluctuations.

In a real experiment we should be estimating certainty intervals for each data point. If it turns out that the intervals are large enough that the second degree polynomial fit through each of them then the this would be a good choice. However, should there be a system where the measurements were quite precise, then indeed the fluctuations might be part of the model and so the fourth order approximation may be better. Compare the two plots below where the measurement precisions correspond to the two cases just discussed.
If additional measurements could be made, a good strategy would be to take these new measurements for velocities where the polynomials calculate very different stresses. Comparing the two polynomials, velocities of .25 or 2.75 seem like good candidates.

There is no degree of polynomial that is the best to use in every case, it depends on the system being modeled as well as the quantity and quality of data. As we have seen, the most complicated is not necessarily the best.

Sample MATLAB Code

```matlab
v=[.5 1 1.5 2 2.5]';
tau= [.75 1.6 2.3 2.32 3.1]';

A_2=[v.^2 v [1;1;1;1;1]];  % second degree polynomial coefficients
A_4=[v.^4 v.^3 v.^2 v [1;1;1;1;1]];  % fourth degree polynomial coefficients

coefficients_2=A_2/tau;
coefficients_4=A_4/tau;

interval=0:.01:3;
quadratic_fit=polyval(coefficients_2,interval);
quartic_fit=polyval(coefficients_4,interval);

figure (1)
plot(v,tau,'x', interval,quartic_fit,'r')
title('Fourth Degree Polynomial Approximation')
axis([0 3 0 5])
figure (2)
plot(v,tau,'x', interval,quadratic_fit,'b')
title('Second Degree Polynomial Approximation')
axis([0 3 0 5])
```
Minimum Norm Problem: Docking two vehicles

Suppose two unit masses move on a line towards each other to dock together. They can make an impulsive maneuver – the velocity instantaneously increases by the amount of the input – at one second intervals from time 0 until and including the final time \( T=4 \). Suppose they start 1 unit distance apart. We can model these simplified dynamics as:

\[
\begin{bmatrix}
  x_1(t+1) \\
  x_2(t+1)
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1(t) \\
  x_2(t)
\end{bmatrix} +
\begin{bmatrix}
  1 \\
  1
\end{bmatrix} u_x(t),
\begin{bmatrix}
  y_1(t+1) \\
  y_2(t+1)
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  y_1(t) \\
  y_2(t)
\end{bmatrix} +
\begin{bmatrix}
  1 \\
  1
\end{bmatrix} u_y(t)
\]

\[
\begin{bmatrix}
  x_1(0) \\
  x_2(0)
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  0
\end{bmatrix},
\begin{bmatrix}
  y_1(0) \\
  y_2(0)
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  0
\end{bmatrix}
\]

\( x_1, x_2 \) are the position and velocity of the first mass and \( y_1, y_2 \) are the position and velocity of the second mass. Do you agree with these equations? In order to dock and not crash into each other, we require that the final positions and velocities be identical.

\[
\begin{align*}
  x_1(0) &= 0 \\
  x_2(0) &= 1 \\
  y_1(0) &= 1 \\
  y_2(0) &= 0
\end{align*}
\]

**Part 1:** Write these two requirements as linear equations.

**Part 2:** Find the inputs that minimize the total input energy \( \sum_{t=0}^{4} u_x^2(t) + \sum_{t=0}^{4} u_y^2(t) \). Draw the resulting velocities over each interval, and make sure the area under the curves means the final positions are the same with identical velocity.

**Part 3:** Suppose instead of minimizing input energy, we wanted to minimize the 1-norm \( \sum_{t=0}^{4} |u_x(t)| + \sum_{t=0}^{4} |u_y(t)| \). Can you find a set of impulses with better performance according to this new norm, or is the result above the minimum for this norm as well?
Solution

Part 1:

We'll start with the velocity requirement \( x_2(T) = y_2(T), \quad T=4. \) Since the changes to velocity are exactly equal to the input amount, we can write the final velocities in terms of the inputs and pose the equality as

\[
\begin{align*}
    u_x(0) + u_x(1) + u_x(2) + u_x(3) + u_x(4) &= u_y(0) + u_y(1) + u_y(2) + u_y(3) + u_y(4) \\
    u_x(0) + u_x(1) + u_x(2) + u_x(3) + u_x(4) - u_y(0) - u_y(1) - u_y(2) - u_y(3) - u_y(4) &= 0
\end{align*}
\]

or

\[
\begin{align*}
    u_x(0) + u_x(1) + u_x(2) + u_x(3) + u_x(4) &= 4u_y(0) + 3u_y(1) + 2u_y(2) + u_y(3) + 0u_y(4) \\
    4u_x(0) + 3u_x(1) + 2u_x(2) + u_x(3) - 4u_y(0) - 3u_y(1) - 2u_y(2) - 1u_y(3) &= 1
\end{align*}
\]

Position is a bit trickier. Note that the resulting change in position due to an impulse is equal to the magnitude multiplied by the time remaining until the final time \( T=4. \) As a result, we can write \( x_1(T) = y_1(T), \quad T=4 \) as

\[
\begin{align*}
    4u_x(0) + 3u_x(1) + 2u_x(2) + u_x(3) + 0u_x(4) &= 4u_y(0) + 3u_y(1) + 2u_y(2) + u_y(3) + 0u_y(4) \\
    4u_x(0) + 3u_x(1) + 2u_x(2) + u_x(3) - 4u_y(0) - 3u_y(1) - 2u_y(2) - 1u_y(3) &= 1
\end{align*}
\]

Part 2:

Writing the two linear equations above as a system in matrix form we get an \( Au=b \) form with \( A \) wide and full rank.

\[
\begin{bmatrix}
    4 & 3 & 2 & 1 & 0 & -4 & -3 & -2 & -1 & 0 \\
    1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix} \begin{bmatrix}
    u_x(0) \\
    u_x(1) \\
    u_x(2) \\
    u_x(3) \\
    u_x(4) \\
    u_y(0) \\
    u_y(1) \\
    u_y(2) \\
    u_y(3) \\
    u_y(4)
\end{bmatrix} = \begin{bmatrix}
    1 \\
    0
\end{bmatrix}
\]

Minimizing the input energy so defined is the same as minimizing the Euclidean norm of the input vector (why?). Using the minimum norm formula \( u_{\text{min}} = A^T (AA^T)^{-1} b \) gives us

\[
u_{\text{min}} = [1.1 \ 0.05 \ 0 -0.05 \ -0.1 \ -0.05 \ 0 \ 0.05 \ 0.1]^T
\]
Plotting the resulting velocities and positions, we see this is indeed a solution.

**Part 3:**

Consider the set of impulses $u_{\text{new}} = \begin{bmatrix} .25 & 0 & 0 & 0 & -0.25 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ in which the first mass approaches the stationary second mass with constant speed and then stops. Verify that indeed $A u_{\text{new}} = b$. We can see that $\|u_{\text{min}}\|_2 = .2236 < .3536 = \|u_{\text{new}}\|_2$ but $\|u_{\text{min}}\|_1 = 1.6 > .5 = \|u_{\text{new}}\|_1$. So when calling something a “minimum” it is important to state what exactly is being minimized.
III. LINEAR TIME INVARIANT (LTI) ODEs

Volume Preserving Flows

For a linear time invariant system given by $\dot{x} = Ax$, we know that $\phi(x_0, t) = e^{At}x_0$ defines the flow. Suppose we would like to know if a system have a volume preserving flow. Among other reasons why this may be useful is that volume preservation in phase space is a characteristic of physical systems with conservation of energy. If $H \subset \mathbb{R}^n$ is a set of points, let $H(t) = \{e^{At}h : h \in H\}$ be the image of $H$ at time $t$. Our flow will be volume preserving if $\text{vol}(H) = \text{vol}(H(t))$ for all $t$.

Part 1: Consider the fact that for arbitrary matrix $M$ and set $S$, $\text{vol}(M(S)) = \det(M)\text{vol}(S)$ where $M(S) = \{Ms : s \in S\}$ is the image of $S$ under $M$. Using this and properties of matrix exponentials to derive a restriction on the system dynamics matrix $A$.

Part 2: Can such a system be asymptotically stable? Can it be stable at all?

Solution

Part 1: $e^{At}$ is itself just a matrix, so from the definition of $H(t)$ and the fact relating determinants to a matrix's effect on volume we must have $\text{vol}(H) = \text{vol}(H(t)) = \det(e^{At})\text{vol}(H)$ for all $t$. As a result $\det(e^{At}) = 1$ is a necessary condition in order to be volume preserving. From properties of matrix exponentials $1 = \det(e^{At}) = e^{tr(At)}$ which means $tr(At) = 0$ for all $t$ and so $tr(A) = 0$.

Part 2: The trace of a matrix, aside from being the sum of its diagonal elements, is also equal to the sum of its eigenvalues. Thus all of the eigenvalues of $A$ have to sum to 0.

If the system were asymptotically stable, all of the eigenvalues would have negative real parts. However this would mean the sum of all the eigenvalues would have negative real part as
well. This means the system cannot be volume preserving. This makes intuitive sense because we can view space as contracting or spiraling in towards the origin when the system is asymptotically stable, shrinking volumes.

Stability is possible when the eigenvalues of $A$ come in conjugate pairs of imaginary numbers, trivially including 0. An example is the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ whose exponential is a rotation matrix.
Chain Reaction

Some simple chemical reactions can be modeled as linear systems. Assume we are dealing with three rather creatively named chemicals: Chemical 1, Chemical 2, and Chemical 3. Let \( x_i \) represent the concentration of Chemical \( i \). Suppose we have a two-step reaction given by:

\[
\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t), \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

Part 1: Compute the eigenvalues and eigenvectors of the linear dynamics matrix by hand. Find the combination of these eigenvectors that gives the initial condition. Use this to write the solution to this initial value problem using the superposition principle.

Part 2: Determine the solution to the initial value problem by calculating the matrix exponential by hand (Hint: Use some of the work from Part 1 to avoid using the definition).

Part 3: Using Matlab, plot the concentrations of each Chemical for 10 seconds.

Part 4: Since this supposed to be modeling concentrations (percentages), what is a condition we should check that the solution satisfies? Is it satisfied in this case? What about the system model makes this happen?

**Solution**

Part 1

\[
0 = det \begin{bmatrix} \lambda+2 & 0 & 0 \\ -2 & \lambda+1 & 0 \\ 0 & -1 & \lambda \end{bmatrix} = (\lambda+2)(\lambda+1)(\lambda)
\]

since the matrix is lower triangular. This means the eigenvalues are 0, -1, -2. Let's calculate the eigenvectors by finding vectors in the null space of \( \lambda I - A \) for each eigenvalue.
\( \lambda_1 = 0 \):

\[
0 = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -2x_1 + x_2 \\ -x_2 \end{bmatrix}.
\]

Clearly we must have \( x_1 = x_2 = 0 \), so take \( v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

\( \lambda_2 = -1 \):

\[
0 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \\ -x_2 - x_3 \end{bmatrix}.
\]

We must have \( x_1 = 0, x_2 = -x_3 \) so take \( v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \).

\( \lambda_3 = -2 \):

\[
0 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2x_1 - x_2 \\ -x_2 - 2x_3 \end{bmatrix}.
\]

We have \( x_2 = -2x_1, x_2 = -2x_3 \) so take \( v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \).

To find the linear combination of eigenvectors that equals the initial condition, want to solve for coefficients such that

\[
c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

This is the same as solving the system

\[
\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

where the columns are the eigenvectors (make sure you understand why this is the case). It is easy to solve by back-substituting to get

\[
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.
\]

This means that from the principle of superposition of system modes we have the solution to the initial value problem is:

\[
x(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^0 + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ 2e^{-t} - 2e^{-2t} \\ 1 - 2e^{-t} + e^{-2t} \end{bmatrix}.
\]

\[
\text{Part 2}
\]

The eigenvalues and eigenvectors have already been calculated, so the easiest way to compute the matrix exponential is to diagonalize and use properties of the exponential.
\[
e^{At} = Q e^{At} Q^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} \exp \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \right) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} e^0 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} e^{-2t} \\ 2e^{-t} - 2e^{-2t} \\ e^{-2t} - 2e^{-t} + 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ e^{-t} & 0 \\ 1 - e^{-t} & 1 \end{bmatrix}
\]

Using the exponential, the solution to the initial value problem is, as expected from Part 1:

\[
x(t) = \begin{bmatrix} e^{-2t} \\ 2e^{-t} - 2e^{-2t} \\ e^{-2t} - 2e^{-t} + 1 \end{bmatrix}
\]

Part 3:

Since we have an analytical formula for the solution, the plot could either be made by integrating and plotting the results, or just plotting the individual variables of the analytical versus time. The code snippet below takes the latter approach.

```matlab
C=[-2 0 0; 2 -1 0; 0 1 0];
t=0:.1:10;
x=[t;t;t];
for i=1:101
x(:,i)=expm(t(i)*C)*[1 0 0]';
end
```
Part 4

Since we are dealing with concentrations of chemicals, they should all sum to 100% for all time. For our example, $x_1(t) + x_2(t) + x_3(t) = e^{-2t} + 2e^{-t} - 2e^{-2t} + e^{-2t} - 2e^{-t} + 1 = 1$ as hoped. In general, the sum of the concentrations is constant because we have:

$$\frac{d}{dt}(x_1(t) + x_2(t) + x_3(t)) = x_1'(t) + x_2'(t) + x_3'(t) = (-2x_1(t)) + (2x_1(t) - x_2(t)) + x_2(t) = 0$$

This is a basic example of a system with an integral of motion which is important in nonlinear systems as we'll see later.

```matlab
figure
hold on
plot(t,x(1,:), 'g')
plot(t,x(2,:), 'c')
plot(t,x(3,:), 'b')
hold off
```
IV. LINEAR TIME VARYING (LTV) ODEs

It seems like this should work: Eigenvalues in LTV systems

It seems like a good idea for LTV system that shows that all of the eigenvalues of the dynamics matrix have negative real part for all time. This would seem to guarantee stability, and indeed you might find this in a couple of engineering papers.

Consider this example, by D.M. Grobman* (of the Hartman-Grobman Theorem we'll see later in the class)


\[ \dot{x} = A(t)x, \quad x \in \mathbb{R}^2 \]

\[ A(t) = \begin{bmatrix} 1 - 4 \cos^2 2t & 2 + \sin 4t \\ -2 + 2 \sin 4t & 1 - 4 \sin^2 2t \end{bmatrix} \]

**Part 1:** Compute the eigenvalues of \( A \) for all time. If this were an LTI system would the system be stable or unstable?

**Part 2:** Verify that \( x(t) = \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix} \) is a solution of the system. (Hint: Use the fact \( \sin 4t = 2 \sin 2t \cos 2t \), which is just a restatement of the \( \sin 2\theta = 2 \sin \theta \cos \theta \) identity)

**Part 3:** Plot either the \( x_1 \) or \( x_2 \) coordinate as a function of time. Is the system stable or unstable?

**Solution:**

**Part 1:**

\[
\text{det} (A(t) - \lambda I) = (1 - 4 \cos^2 2t - \lambda)(1 - 4 \sin^2 2t - \lambda) - (-2 + \sin 4t)(2 + 2 \sin 4t)
\]

\[
= (1 - 4 \cos^2 2t)(1 - 4 \sin^2 2t - \lambda) - (1 - 4 \cos^2 2t + 1 - 4 \sin^2 2t) + \lambda^2 - (4 + 4 \sin^2 4t)
\]

\[
= \lambda^2 + 2\lambda + (1 - 4 \sin^2 2t - 4 \cos^2 2t + 16 \cos^2 2t \sin^2 2t + 4 - 16 \sin^2 2t \cos^2 2t)
\]

\[
= \lambda^2 + 2\lambda + 1
\]
As a result, the system has a repeated eigenvalue of -1 for all time. This is an interesting example because not only are the eigenvalues negative for all time, they are constant. If any LTV system should be stable looking at the eigenvalues of $A(t)$ it seems like it would be this one.

Part 2:

$$A(t)x(t)=\begin{bmatrix} 1-4\cos^2 2t & 2+\sin 4t \\ -2+2\sin 4t & 1-4\sin^2 2t \end{bmatrix} \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}$$

$$=e^t \begin{bmatrix} \sin 2t-4\cos^2 2t \sin 2t+2\cos 2t+2\sin 4t \cos 2t \\ -2\sin 2t+2\sin 4t \sin 2t+\cos 2t-4\sin^2 2t \cos 2t \end{bmatrix}$$

$$=e^t \begin{bmatrix} \sin 2t-4\cos^2 2t \sin 2t+2\cos 2t+4\sin 2t \cos^2 2t \\ -2\sin 2t+4\sin^2 2t \cos 2t+\cos 2t-4\sin^2 2t \cos 2t \end{bmatrix}$$

$$=e^t \begin{bmatrix} \sin 2t+2\cos 2t \\ -2\sin 2t+\cos 2t \end{bmatrix} = \begin{bmatrix} e^t \sin 2t+2e^t \cos 2t \\ e^t \cos 2t-2e^t \sin 2t \end{bmatrix} = \dot{x}(t)$$

Part 3:

The following plots the $x_1$ coordinate for 10 seconds. It is clearly unstable, so is an example why LTI style eigenvalue analysis is abandoned in favor of other methods like Lyapunov Exponents and Floquet Theory for periodic systems.

```matlab
t=0:.01:10;
x=exp(t).*sin(2*t);
figure
plot(t,x)
axis([0 10 -1000 1000])
```
Calculating a State Transition Matrix

Consider the LTV system given by

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix} 0 & t \\
0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\
x_2(t) \end{bmatrix}.
\]

Calculate the State Transition Matrix by first finding the Fundamental Matrix. Compare this to the matrix exponential of \( A(t) \) even when \( t_0 = 0 \).

Solution

This is equivalent to the system \( \dot{x}_1(t) = t \ x_2(t) \), \( \dot{x}_2(t) = 0 \). If \( x_1(0) = 1, x_2(0) = 0 \) then we would have \( x_2(t) = 0 \) for all time and so \( x_1(t) = 1 \). If \( x_1(0) = 0, x_2(0) = 1 \) then \( x_2(t) = 1 \) for all time and so \( \dot{x}_1(t) = t \) and so \( x_1(t) = \frac{t^2}{2} \).

The \( i \)-th column of a matrix in a given basis is equal to the whole matrix multiplied by the \( i \)-th basis vector. The considerations above written in matrix form tell us that the Fundamental Matrix, which takes \( t_0 = 0 \) initial conditions to their solution trajectories, satisfies

\[
\Psi(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \Psi(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t^2/2 \\ 1 \end{bmatrix}
\]

and so \( \Psi(t) = \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \). Verifying this is a Fundamental Matrix, \( \dot{\Psi}(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \) and so indeed the derivative condition is satisfied.

This means the State Transition matrix is

\[
\Phi(t, t_0) = \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & t_0^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & (t^2 - t_0^2)/2 \\ 0 & 1 \end{bmatrix}
\]

To compute the matrix exponential, first note that \( A^2 = A^3 = A^4 = \ldots = 0 \). This means our definition reduces to \( I + tA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \). The upper right value is missing the \( \frac{1}{2} \) term that the State Transition Matrix has.
V. NONLINEAR ALGEBRAIC EQUATIONS

Continuing Example: Circular Restricted Three-Body Problem – Equilibrium Point

Calculating the position of the colinear (L1, L2, L3) equilibrium points involves deriving and solving a complicated nonlinear algebraic equation. Do Problem 3 of the PCR3BP section in Continuing Examples.

Terminal Velocity

A flat plate is falling freely through the air subject to the force of gravity and drag. The force of drag (upward) is given by
\[
f_D = \frac{0.3|v|^2}{500 + (\log|v|)^3} - 0.02|v| \quad \text{where } v \text{ is the velocity in m/s.}
\]
The mass of the plate is 5 kg. Set up a problem of the form \( F(v) = 0 \) and solve for the terminal velocity. Also, plot the velocity starting from \( v=0 \) and see how this compares to the algebraic result.

Solution

Terminal velocity is achieved when the force of gravity and drag are equal in magnitude. This means we are trying to solve
\[
F = f_D - mg = \frac{0.3|v|^2}{500 + (\log|v|)^3} - 0.02|v| - 5 \times 9.8 = 0
\]

function terminalvelocity()
%******** Integrate to plot velocity ********
time_span=[0 30];
v0=0;
[t,velocity] = ode45(@calc_velocity, time_span, v0);
figure(1)
plot(t,velocity,'b')
%

%******** Use Nonlinear Solver to Calculate Terminal Velocity *****
initial_guess=-200;
options=optimset('Display','iter'); % Option to display output
[x,fval] = fsolve(@myfun,initial_guess,options) % Call solver
end

function F = myfun(v)
F = (.3*abs(v)^2)/(500+ (log(abs(v)))^3)-.02*abs(v)-9.8*5;
end

function dy = calc_velocity(t,v)
dy = (.3*abs(v)^2)/(500+ (log(abs(v)))^3)-.02*abs(v)-9.8*5;
end
The plot of the velocity over time is:

If the initial guess used to solve the algebraic equation did not lead to convergence, using the plot for a better initial guess is a good idea. As it is, -200 was good enough in this case.

The output from fsolve is

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Func-count</th>
<th>f(x)</th>
<th>Norm of step</th>
<th>First-order optimality</th>
<th>Trust-region radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1190.42</td>
<td>5.28</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1179.86</td>
<td>5.28</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>1153.46</td>
<td>5.28</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>1087.42</td>
<td>5.28</td>
<td>6.25</td>
<td>6.25</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>923.257</td>
<td>5.21</td>
<td>15.6</td>
<td>15.6</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>535.533</td>
<td>4.61</td>
<td>39.1</td>
<td>39.1</td>
</tr>
<tr>
<td>6</td>
<td>14</td>
<td>0.170346</td>
<td>0.11</td>
<td>97.7</td>
<td>97.7</td>
</tr>
<tr>
<td>7</td>
<td>16</td>
<td>6.34956e-07</td>
<td>0.000212</td>
<td>244</td>
<td>244</td>
</tr>
<tr>
<td>8</td>
<td>18</td>
<td>8.65768e-18</td>
<td>7.84e-10</td>
<td>244</td>
<td>244</td>
</tr>
</tbody>
</table>

Equation solved.

\[ x = -363.6448 \]

\[ fval = 2.9424e-09 \]

A terminal velocity of 363.6448 m/s down seems to be the correct result and both methods compare well.
A Couple of Orbital Elements

Consider the following algebraic equations relating the position $\vec{r}=(r_1, r_2, r_3)$ and velocity $\vec{v}=(v_1, v_2, v_3)$ in the classical model of a small body rotating near a planet or star.

**Part 1:** For these orbits we have $\vec{h}=\vec{r}\times\vec{v}=constant$. Explain how this shows that the satellite motion is confined to a plane. Also, if $\vec{h}$ is known exactly, show how this and the fact motion is confined to a plane can be used to define a state using three out of the four following pieces of information: the angle of $\vec{r}$ relative to some reference direction on the plane, the magnitude of $\vec{r}$, the angle of $\vec{v}$ relative to $\vec{r}$, and the magnitude of $\vec{v}$.

**Part 2:** For these orbits we also have $E(\vec{r}, \vec{v})=\frac{1}{2}v^2-\frac{1}{r}=constant$. Continuing the scheme above, state how the state may be completely described knowing $\vec{h}, E$, and the angle and magnitude of the position vector.

**Part 3:** There are other equations like the above (examples of integrals of motion to be discussed in the next section), but even with these two sets of equations we can solve for the distance and magnitude of velocity at the closest and farthest points of the orbit. Assume $h=[0\ 0\ 1.2]$ and $E=-0.28$ Also, if in addition to this information we only knew that the angle between the x-axis and the closest point is 0, what would the states be of the closest and farthest point. Using these distances, also calculate the eccentricity given by $\frac{r_{\text{max}}-r_{\text{min}}}{r_{\text{max}}+r_{\text{min}}}$.

**Solution**

**Part 1:**

At some initial time we have that $\vec{r}_0, \vec{v}_0$ span a plane that has $\vec{h}=\vec{r}_0 \times \vec{v}_0$ as a normal vector. $\vec{h}=\vec{r}\times\vec{v}$ always defines a normal vector to the plane containing $\vec{r}, \vec{v}$ and since $\vec{h}$ is constant, $\vec{r}, \vec{v}$ must always lie on the same plane.

Since $\vec{r}, \vec{v}$ lie on a plane, once this plane is specified by $\vec{h}$ we can define $\vec{r}$ by an angle relative to a reference direction and its magnitude. Essentially this is a form of polar coordinates. Similarly we can define $\vec{v}$ by an angle relative to $\vec{r}$ and its magnitude. At this point in time by specifying the plane we have reduced the system to four independent
coordinates needed to define a state. But \( \vec{h} \) not only defines a plane, it also gives us a relationship \( \|\vec{h}\| = \|\vec{r}\| \|\vec{v}\| \sin \theta \) where \( \theta \) is the angle between the two. Thus if we know \( \vec{h} \), we can further reduce the above scheme to three independent coordinates by using \( \|\vec{h}\| = \|\vec{r}\| \|\vec{v}\| \sin \theta \) to calculate the magnitude of \( \vec{v} \), once the direction and magnitude of \( \vec{r} \), and angle between \( \vec{r}, \vec{v} \), are specified. Other permutations exist such as calculating the angle between \( \vec{r}, \vec{v} \) if \( |\vec{v}| \) were known instead.

Solving this vector equation, which is really solving three scalar equations

\[
\begin{bmatrix}
r_2v_3 - r_3v_2 \\
r_3v_1 - r_1v_3 \\
r_1v_2 - r_2v_1
\end{bmatrix}
= \begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
\]

allows us to reduce the system dimension by three.

**Part 2:**

Given the angle and magnitude of the position in the plane normal to \( \vec{h} \), we can calculate \( |\vec{v}| = \sqrt{2E + \frac{2}{r}} \) using the formula for \( E \). The angle between the two vectors can then be calculated using \( \pm|\vec{h}| = |\vec{r}| |\vec{v}| \sin \theta \).

**Part 3:**

At both closest and farthest points of the orbit, the velocity must be perpendicular to the position vector, otherwise a component of the velocity would be in the direction of the position vector and so neighboring points would exist with larger and smaller distances. Given that \( \vec{h} \) has a positive \( z \)-component we know the satellite must be moving counterclockwise if viewed from above. \( |\vec{h}| = |\vec{r}| |\vec{v}| \sin \theta \) reduces to \( 1.2 = rv \) in these two cases, which we can rewrite as

\[
\frac{1}{r} = \frac{v}{1.2}.
\]

Plugging this into the formula for \( E \) we get \( \frac{1}{2} v^2 - \frac{1}{r} = \frac{1}{2} v^2 - \frac{v}{1.2} = E \). We can then find \( v \) by solving \( \frac{1}{2} v^2 - \frac{5}{6}v + .28 = 0 \) to get \( v = 1.2, 0.4667 \) at the extrema. Rearranging the equation for \( E \) again we solve \( r = \frac{1}{\frac{1}{2} v^2 - E} \). For \( v = 1.2 \) we get \( r = 1 \), while for \( v = 0.4667 \) we get \( r = 2.5713 \). Taking the x-axis to point towards the closest point the states are: \((1, 0, 0, 0, 1.2, 0)\) and \((2.5713, 0, 0, 0, -0.4667, 0)\). Not bad for not using any equations of motion.

The eccentricity is then .44.
VI. NONLINEAR ODES

Van der Pol Oscillator

The Van der Pol oscillator is defined by the equation \( \ddot{x} - \mu (1-x^2) \dot{x} + x = 0, \mu \geq 0 \). It was originally developed to model circuits with vacuum tubes, but has been used in other physical applications such as action potential in neurons. Note that this is similar in form to the mass-spring-damper system, but with a nonlinear term on what was the damping term. If \( \mu = 0 \) it is simply a linear mass-spring system.

Part 1: Put the system in first order form.

Part 2: Linearize the system around the equilibrium point at the origin. Is the origin stable or unstable? Classify the origin in the linear dynamics (source, sink, etc.).

Part 3: For each value \( \mu = 0, 0.5, 1, \) and \( 3 \) make a plot of the two trajectories given by initial conditions \((0, -0.5)\), and \((1, -2)\). What features are visible that are not possible in a linear system?

Solution

Part 1: \[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\end{bmatrix} = \begin{bmatrix}
x_2 \\
\mu(1-x_1^2)x_2-x_1 \\
\end{bmatrix}
\]

Part 2: The top row is trivial, and for the second row we have \( \frac{\partial}{\partial x_1}(\mu(1-x_1^2)x_2-x_1) = 2\mu x_1 x_2 - 1 \) and \( \frac{\partial}{\partial x_2}(\mu(1-x_1^2)x_2-x_1) = \mu(1-x_1^2) \) Evaluating all of the partial derivatives at the origin gives the linearized dynamics \[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & \mu \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix} .
\]
The characteristic equation is \( 0 = \lambda(\lambda - \mu) + 1 = \lambda^2 - \mu \lambda + 1 \). This results in eigenvalues \( \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} \). This gives us the following cases:
\( \mu = 0 \): The eigenvalues are purely imaginary, meaning the origin is a center. This is to be expected since this value of \( \mu \) leads to a simple harmonic oscillator.

\( 0 < \mu < 2 \): The eigenvalues are complex, but have positive real part. This means the origin is an unstable focus/spiral.

\( \mu \geq 2 \): Both eigenvalues are positive, so the origin is a source.

**Part 3:**

In linear systems, periodic orbits only exist when the origin is a center (Why is this? Think about writing any trajectory in a linear system in terms of the modes/eigenvectors of the system). When \( \mu > 0 \) we saw that the origin is either an unstable focus or a source, however there appears to be a periodic orbit despite this. Another difference is that in this nonlinear system, the periodic orbit is attractive, which is not possible in linear systems. A periodic orbit that is the limit for other trajectories is called a *limit cycle*. 
**Continuing Example: Circular Restricted Three-Body Problem – Linearized Stability**

Do Problem 4 of the PCR3BP Example in the Continuing Examples section. This will determine the stability of the linearized system near the L1 equilibrium point and holds for L2 and L3 as well. Several missions focus on these points so the design is dependent on this result.

**Continuing Example: Circular Restricted Three-Body Problem – Applying Manifolds**

Do Problem 5 of the PCR3BP Example in the Continuing Examples section. The seemingly theoretical invariant manifolds play a practical role in enabling fuel efficient transfers.

**Continuing Example: Circular Restricted Three-Body Problem – Forbidden Realms**

Do Problem 6 of the PCR3BP Example in the Continuing Examples section. An integral of motion tells us where a spacecraft can and cannot go.

**Continuing Example: ODE with Parameter Bifurcation – Every trick in the book**

Getting a complete picture of the stability characteristics of this family of systems will require the use of linearization, integrals of motion, and Lyapunov functions. Do Problem 2, 3, and 4 of the example in the Continuing Examples section.
Continuing Example: The Planar Circular Restricted Three Body Problem

The Circular Restricted Three Body Problem (CR3BP) is a very common and important model for a satellite moving under the influence of two massive bodies rotating around their combined center of mass, usually with one much larger than the other. Examples of this would be a satellite exploring the Earth/Moon, Jupiter/Europa, or even the Sun/Earth system ignoring smaller effects of other planets and moons. Other restrictions include assuming that the massive bodies move in a circular rather than elliptical orbit around the center of mass. A common subproblem is to restrict the motion of the satellite to the orbital plane of the massive bodies, which is the Planar Circular Restricted Three Body Problem (PCR3BP).

Equations of Motion

Rather than use the equations of motion in an inertial frame, equations are derived to describe motion in the rotating frame that follows the rotation of the massive bodies. In the diagram below this means we are dealing with the coordinates $x,y$ rather than the fixed $X,Y$. The two bodies $m_1, m_2$ and the rotating frame that follows them move with period $2\pi$ around the origin (units of time have been scaled to achieve this). Notice that in the rotating frame the massive bodies appear fixed.

![Diagram of the CR3BP](image)

The equations of motion in the rotating frame described above are

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= -\frac{d\hat{U}}{dx} \\
\ddot{y} + 2\dot{x} &= -\frac{d\hat{U}}{dy}
\end{align*}
\]
where $\tilde{U}(x, y) = -\frac{1}{2}(x^2 + y^2) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} - \frac{1}{2} \mu_1 \mu_2$ is the Effective Potential. $\mu_2$ is the ratio of the masses of the two bodies, so for example in the Jupiter-Europa system $\mu_2 = \frac{\text{mass} \text{Europa}}{\text{mass} \text{Jupiter}} = 2.528 \times 10^{-5}$. $\mu_1 = 1 - \mu_2$, and $r_1, r_2$ are the distances from the satellite to the primary and secondary bodies. These are given by $r_1^2 = (x + \mu_2)^2 + y^2$ and $r_2^2 = (x - \mu_1)^2 + y^2$. Details of the derivation may be found in several books, including the book *Dynamical Systems, the Three-Body Problem and Space Mission Design* by Koon, Lo, Marsden, and Ross which is available for free on Shane Ross' website.

It is a fair question to ask why mission designers use these equations. In many missions the position of the satellite relative to the two bodies is the most important feature. For example, while approaching the Moon it is much easier to define distance to the point that represents the moon in the PCR3BP than constantly changing coordinates in an inertial frame as the Moon whirs around. Another reason is that some trajectories that look unremarkable in a fixed frame reveal a lot more structure in the rotating frame. Below is the path of the real comet Oterma in the Sun/Jupiter system in both frames.

**Problem 1: Convert to first order form**

Put the equations of motion into first order form. Leave the $\frac{d \tilde{U}}{dx}, \frac{d \tilde{U}}{dy}$ terms as-is for now. In MATLAB, try to plot a few different initial conditions. MATLAB doesn't mind messy terms, so feel free to paste in results from differentiating symbolically for the derivatives.
Solution

We will use coordinates $x, y, v_x, v_y$ rather than $x_1, x_2, x_3, x_4$ to keep the relationships between positions and their velocities a bit more clear.

$$
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{v}_x \\
\dot{v}_y
\end{bmatrix}
= 
\begin{bmatrix}
v_x \\
v_y \\
2v_y \frac{d \bar{U}}{dx} \\
-2v_x \frac{d \bar{U}}{dy}
\end{bmatrix}
$$

Problem 2: Setting Error Tolerances in Matlab

Integrate the following PCR3BP initial condition in Matlab:

$x(0)=.9, y(0)=0, v_x(0)=-.1, v_y(0)=.03$. Create trajectories using the following error tolerances:

<table>
<thead>
<tr>
<th>Relative</th>
<th>Absolute</th>
<th>Color</th>
</tr>
</thead>
<tbody>
<tr>
<td>1e-2</td>
<td>1e-1</td>
<td>Yellow</td>
</tr>
<tr>
<td>1e-3</td>
<td>1e-2</td>
<td>Green</td>
</tr>
<tr>
<td>1e-4</td>
<td>1e-3</td>
<td>Cyan</td>
</tr>
<tr>
<td>1e-5</td>
<td>1e-4</td>
<td>Blue</td>
</tr>
<tr>
<td>1e-6</td>
<td>1e-5</td>
<td>Black</td>
</tr>
</tbody>
</table>

Plot all five trajectories together in the same figure in the rotating frame of the PCR3BP. Also, try to plot these in an inertial frame as well. To do this, use the conversion

$$
X_{\text{inertial}}(t) = \cos(t) x(t) - \sin(t) y(t)
$$

$$
Y_{\text{inertial}}(t) = \sin(t) x(t) + \cos(t) y(t)
$$

Looking at these plots, for this initial condition and timescale would there be benefits for increasing the integration accuracy even more?

Solution

Aside from using different order integrators like ode23 and ode45, specific relative and absolute error tolerances can be set using 'options' in Matlab. The format of setting options is shown below, in addition how to use them in the integrator:
Here are the plots in the rotating and inertial frame, respectively:

By zooming in on either of the plots, we can see the increasing accuracy makes the trajectories converge towards what we hope to be the true solution. In fact, the fourth and fifth sets of tolerances (in blue and black) are very hard to distinguish. This is usually a sign the tolerances are sufficiently accurate; if the differences between results is not shrinking, better tolerances may be needed.
function CR3BP()
%******** Define Parameters ********
% Europa
mu2 = 2.528*10^-5; % Mass Parameter of Secondary Body
mu1 = 1-mu2; % Mass Parameter of Primary Body
radius_1 = 71492; % Radius of Primary Body in km. Used for plot only.
radius_2 = 1569; % Radius of Secondary Body in km. Used for plot only.
distance_1_2 = 671100; % Center to center distance from Primary to Secondary in Circular Model. Plot only.

time_span=[0 20]; % in time units scaled such that secondary body period of orbit is 2*pi
x0=[.9 0 0 -.1 .03 0]
%
%******** Integrate Trajectory ********
% Customized Tolerances
options_1=odeset('RelTol',1e-2,'AbsTol',1e-1);
options_2=odeset('RelTol',1e-3,'AbsTol',1e-2);
options_3=odeset('RelTol',1e-4,'AbsTol',1e-3);
options_4=odeset('RelTol',1e-5,'AbsTol',1e-4);
options_5=odeset('RelTol',1e-6,'AbsTol',1e-5);
[t_1, x_1] = ode45(@CR3BP, time_span, x0, options_1);
[t_2, x_2] = ode45(@CR3BP, time_span, x0, options_2);
[t_3, x_3] = ode45(@CR3BP, time_span, x0, options_3);
[t_4, x_4] = ode45(@CR3BP, time_span, x0, options_4);
[t, x] = ode45(@CR3BP, time_span, x0, options_5);
length_t=length(t)
%
% ******** Plot Solution in Rotating Frame ********
figure(1)
hold on
% Plot Primary Body
eradius=radius_1/distance_1_2; % Radius in the scaled units
[xe,ye,ze]=sphere(20);
x=-mu2
ye=radius1*ye;
ze=radius1*ze;
surf(xe,ye,ze);
hold on
% Plot Secondary
mradius=radius_2/distance_1_2; % Radius in the scaled units
[xm,ym,zm]=sphere(10);
xm=mu1
ym=mradius*ym;
z=mradius*zm;
surf(xm,ym,zm);
hold on

% PLOT TRAJECTORIES
plot3(x_1(:,1),x_1(:,2),x_1(:,3),'y');
plot3(x_2(:,1),x_2(:,2),x_2(:,3),'g');
plot3(x_3(:,1),x_3(:,2),x_3(:,3),'c');
plot3(x_4(:,1),x_4(:,2),x_4(:,3),'b');
plot3(x(:,1),x(:,2),x(:,3),'k');
axis('equal');
xlabel('x-axis');
ylabel('y-axis');
axis([-2 2 -2 2 -1 1])
%
% ******** Plot Solution in Fixed Frame ********
figure(2)
X_1_fixed = cos(t_1).*x_1(:,1) - sin(t_1).*x_1(:,2);
\[ Y_1_{\text{fixed}} = \sin(t_1) \cdot x_1(:,1) + \cos(t_1) \cdot x_1(:,2); \]
\[ Z_1_{\text{fixed}} = x_1(:,3); \]
\[ X_2_{\text{fixed}} = \cos(t_2) \cdot x_2(:,1) - \sin(t_2) \cdot x_2(:,2); \]
\[ Y_2_{\text{fixed}} = \sin(t_2) \cdot x_2(:,1) + \cos(t_2) \cdot x_2(:,2); \]
\[ Z_2_{\text{fixed}} = x_2(:,3); \]
\[ X_3_{\text{fixed}} = \cos(t_3) \cdot x_3(:,1) - \sin(t_3) \cdot x_3(:,2); \]
\[ Y_3_{\text{fixed}} = \sin(t_3) \cdot x_3(:,1) + \cos(t_3) \cdot x_3(:,2); \]
\[ Z_3_{\text{fixed}} = x_3(:,3); \]
\[ X_4_{\text{fixed}} = \cos(t_4) \cdot x_4(:,1) - \sin(t_4) \cdot x_4(:,2); \]
\[ Y_4_{\text{fixed}} = \sin(t_4) \cdot x_4(:,1) + \cos(t_4) \cdot x_4(:,2); \]
\[ Z_4_{\text{fixed}} = x_4(:,3); \]
\[ X_{\text{fixed}} = \cos(t) \cdot x(:,1) - \sin(t) \cdot x(:,2); \]
\[ Y_{\text{fixed}} = \sin(t) \cdot x(:,1) + \cos(t) \cdot x(:,2); \]
\[ Z_{\text{fixed}} = x(:,3); \]

\[ \text{eradius} = \frac{\text{radius}_1}{\text{distance}_1_2}; \quad \% \text{Radius in the scaled units} \]
\[
[xe, ye, ze] = \text{sphere}(20);
\]
\[ xe = \text{eradius} * xe - \text{mu2}; \quad \% \text{Shift center to (-mu2, 0, 0)} \]
\[ ye = \text{eradius} * ye; \]
\[ ze = \text{eradius} * ze; \]
\[ \text{surf}(xe, ye, ze); \]
\[ \text{hold on} \]
\[ \text{moon_orbit}_x = \text{mul} * \cos(t); \]
\[ \text{moon_orbit}_y = \text{mul} * \sin(t); \]
\[ \text{moon_orbit}_z = 0 \cdot t; \]
\[ \text{plot3}(\text{moon_orbit}_x, \text{moon_orbit}_y, \text{moon_orbit}_z, '^r'); \]
\[ \text{hold on} \]
\[ \text{plot3}(X_1_{\text{fixed}}, Y_1_{\text{fixed}}, Z_1_{\text{fixed}}, '^y'); \]
\[ \text{plot3}(X_2_{\text{fixed}}, Y_2_{\text{fixed}}, Z_2_{\text{fixed}}, '^g'); \]
\[ \text{plot3}(X_3_{\text{fixed}}, Y_3_{\text{fixed}}, Z_3_{\text{fixed}}, '^c'); \]
\[ \text{plot3}(X_4_{\text{fixed}}, Y_4_{\text{fixed}}, Z_4_{\text{fixed}}, '^b'); \]
\[ \text{plot3}(X_{\text{fixed}}, Y_{\text{fixed}}, Z_{\text{fixed}}, '^k'); \]
\[ \text{axis('equal')}; \]
\[ \text{axis([-2 2 -2 2])}; \]
\[ \text{hold off} \]

\begin{verbatim}
function \ dx = CR3BP(t,x) \end{verbatim}
\[ \text{dx} = \text{zeros}(6,1); \]
\[ \text{dx}(1) = x(4); \]
\[ \text{dx}(2) = x(5); \]
\[ \text{dx}(3) = x(6); \]
\[ \% x ddot = 2ydot - Ux \]
\[ \text{dx}(4) = 2 \cdot x(5) + x(1) - ((\text{mul} - x(1)) \cdot x(3)^2 + x(2)^2 + x(3)^2) / ((\text{mul} + x(1))^2 + x(2)^2 + x(3)^2)^{3/2}; \]
\[ \% y ddot = -2xdot - Uy \]
\[ \text{dx}(5) = -2 \cdot x(4) + x(2) - ((\text{mul} + x(1)) \cdot x(2)^2 + x(2)^2 + x(3)^2) / ((\text{mul} + x(1))^2 + x(2)^2 + x(3)^2)^{3/2}; \]
\[ \% z ddot = - Uz \]
\[ \text{dx}(6) = -(\text{mul} + x(3)) / ((\text{mul} + x(1))^2 + x(2)^2 + x(3)^2)^{3/2} + (\text{mul} + x(3)) / ((\text{mul} - x(1))^2 + x(2)^2 + x(3)^2)^{3/2}); \]
\end{verbatim}

\[ \text{end} \]
\[ \text{end} \]
Problem 3: Calculating an equilibrium point

The PCR3BP has five equilibrium points, frequently called Lagrange or libration points. These are marked below and the specific points are referred to as L1 through L5 as shown below. L1, L2, and L3 lie on the line running through the two massive bodies, while L4 and L5 lie on a vertex of an equilateral triangle connecting the two bodies.

It is important to remember that a fixed point in the rotating frame is actually a circular orbit in the inertial frame with the same angular velocity as the massive bodies. This means, for example, that a satellite at L1 in the Sun/Earth system will always stay the exact same distance on the line between the two. This can be very useful for certain types of missions. The Genesis mission collected particles from the solar wind by staying on an orbit that stayed near the Sun/Earth L1 point for several months. The replacement for Hubble, the James Webb Space Telescope will be located at the L2 point so that the Earth will always be able to shade it from some of the Sun's light pollution.

We will calculate the location of L1 in the Jupiter/Europa system \( \mu_2 = 2.528 \times 10^{-5} \) with the help of MATLAB. Perform the following steps:

**Step 1:** Expand \( \frac{d\hat{U}}{dx} \) out for the case when \( y=0 \). Be careful with this step. Remember that \( \sqrt{(z^2)} = |z| \neq z \).
**Step 2:** Looking at the system in first order form, *at an equilibrium point*, what condition do we have on \( \frac{d \hat{U}}{dx} \)?

**Step 3:** Use the formula from Step 1 and a nonlinear solver in MATLAB such as `fsolve` to find the value of \( x \) such that the condition from step 2 is satisfied. The result, if it converges, will give you the x-coordinate of the equilibrium point. `Fsolve` requires an initial guess, so depending on this initial guess you will find a different equilibrium point. What conditions on this guess will lead to finding L1?

**Step 4:** Verify this value has actually converged to a solution by plotting the effective potential with \( y=0 \) in a small interval near the calculated equilibrium point. A maximum value should occur at this point.

**Solution**

**Step 1:**

\[
\frac{d \hat{U}}{dx} = -x + \frac{\mu_1}{r_1^2} \frac{1}{2} ((x + \mu_2)^2 + y^2)^{-\frac{3}{2}} 2(x + \mu_2) + \frac{\mu_2}{r_2^2} ((x - \mu_1)^2 + y^2)^{-\frac{3}{2}} 2(x - \mu_1)
\]

\[
= -x + \frac{\mu_1}{r_1^2} ((x + \mu_2)^2)^{-\frac{1}{2}} (x + \mu_2) + \frac{\mu_2}{r_2^2} ((x - \mu_1)^2)^{-\frac{1}{2}} (x - \mu_1) \quad \text{at } y=0
\]

\[
= -x + \frac{\mu_1}{r_1^2} \frac{(x + \mu_2)}{|x + \mu_2|} + \frac{\mu_2}{r_2^2} \frac{(x - \mu_1)}{|x - \mu_1|} \quad \text{at } y=0
\]

\[
= -x + \frac{\mu_1}{r_1^2} \text{sgn}(x + \mu_2) + \frac{\mu_2}{r_2^2} \text{sgn}(x - \mu_1) \quad \text{at } y=0
\]

\[
= -x + \frac{\mu_1}{(x + \mu_2)^2} \text{sgn}(x + \mu_2) + \frac{\mu_2}{(x - \mu_1)^2} \text{sgn}(x - \mu_1) \quad \text{at } y=0
\]

**Step 2:**

All of the other terms will be zero, leaving simply \( \frac{d \hat{U}}{dx} = 0 \).

**Step 3:**

`fsolve` needs an initial guess to solve \( \frac{d \hat{U}}{dx} = 0 \). To find L1, a value between 0 and \( \mu_1 \)
will theoretically lead to finding L1 if it converges to an equilibrium point. However, nonlinear solvers often depend on good guesses, so something closer to the smaller massive body is probably better. Below, $x=0.9$ was used.

**Step 4:**

The program below calculates L1 to be at (0.9798...,0) and converged in six iterations. We can see this value matches the extrema of the effective potential restricted to $y=0$.

```matlab
function Calculate_L1()

clear all

mu2=2.528*10^-5; %Mass parameter for the Jupiter/Europa system
mu1=1-mu2;

x0=.9; %Initial guess needed for fsolve, can be anywhere between the two bodies
options=optimset('Display','iter'); %Display steps before convergence
[xe, fval]=fsolve(@myfun, x0, options) %Solve for xe, the equilibrium point

%Plot Effective Potential, visually check that extremum occurs at xe
x=.979:.0001:.981;
y=x;
for i=1:length(x)
    z(i)=-.5*x(i)^2 - mu1/abs(x(i)+mu2) - mu2/abs(x(i)-mu1);
end
```

![Plot of Effective Potential](image.png)
end
figure(1)
plot(t,z)
end

%Function definition used by fsolve.
function F = myfun(x)
mu2=2.528*10^-5;
mu1=1-mu2;
F = -x + mu1/(x+mu2)^2*sign(x+mu2) + mu2/ (x-mu1)^2*sign(x-mu1); %dU/dx when y=0
end
Problem 4: Determining the Linearized Stability of L1

Recall that the first order form of the PCR3BP is

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{v}_x \\
\dot{v}_y
\end{bmatrix} =
\begin{bmatrix}
v_x \\
v_y \\
2v_y - \frac{d\tilde{U}}{dx} \\
-2v_x - \frac{d\tilde{U}}{dy}
\end{bmatrix}
\]

Step 1: Determine the linearized dynamics around L1.

Step 2: What are the eigenvalues of the dynamics matrix for the values of \( \mu, x_e, y_e \)? Does this mean L1 is stable or unstable?

Solution

Step 1:

The linearized dynamics are given by:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{v}_x \\
\dot{v}_y
\end{bmatrix} =
\begin{bmatrix}
v_x \\
v_y \\
2v_y + a\dot{x} \\
-2v_x - b\dot{y}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
a & 0 & 0 & 2 \\
0 & -b & -2 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
v_x \\
v_y
\end{bmatrix}
\]

where \( a = 1 + 2 \frac{\mu_1}{|x_e + \mu_2|^3} + 2 \frac{\mu_2}{|x_e - \mu_1|^3} \) and \( b = -1 + \frac{\mu_1}{|x_e + \mu_2|^3} + \frac{\mu_2}{|x_e - \mu_1|^3} \). We will look into calculating the bottom row of this matrix, the third row is similar.

From the first order form we have that \( \dot{v}_y = -2v_x - \frac{d\tilde{U}}{dy} \). Taking the partial derivative of \( \tilde{U} \) gives

\[
\frac{d\tilde{U}}{dy} = -y + \frac{\mu_1}{r_1^2} \frac{1}{2} ((x + \mu_2)^2 + y^2)^{\frac{1}{2}} 2y + \frac{\mu_2}{r_2^2} \frac{1}{2} ((x - \mu_1)^2 + y^2)^{\frac{1}{2}} 2y = -y + \frac{\mu_1}{((x + \mu_2)^2 + y^2)^{\frac{3}{2}}} y
\]

\[
+ \frac{\mu_2}{((x - \mu_1)^2 + y^2)^\frac{3}{2}} y.
\]

So \( \dot{v}_y = -2v_x - \left( -y + \frac{\mu_1}{((x + \mu_2)^2 + y^2)^{\frac{3}{2}}} y + \frac{\mu_2}{((x - \mu_1)^2 + y^2)^{\frac{3}{2}}} y \right) \).
To fill out the matrix we calculate the partial derivatives of this function with respect to $x, y, v_x, v_y$ and evaluate them at $(x_e, 0, 0, 0)$.

**Column 1** -
\[
\frac{\partial v_y}{\partial x}(x_e,0,0,0) = \frac{\mu_1}{(x+\mu_2)^2 + y^2} \frac{y}{2} (2(x+\mu_2)) + \frac{\mu_2}{(x-\mu_1)^2 + y^2} \frac{y}{2} (2(x+\mu_2)) = 0 \quad \text{when} \quad x = x_e, y = 0.
\]

**Column 2** -
\[
\frac{\partial v_y}{\partial y}(x_e,0,0,0) = 1 - \frac{\mu_1}{((x+\mu_2)^2 + y^2)^{3/2}} - \frac{\mu_2}{((x-\mu_1)^2 + y^2)^{3/2}} + \text{terms } y \quad \text{via the product rule}
\]
\[= 1 - \frac{\mu_1}{((x+\mu_2)^2)^{3/2}} - \frac{\mu_2}{((x-\mu_1)^2)^{3/2}} = 1 - \frac{\mu_1}{|x+\mu_2|^3} - \frac{\mu_2}{|x-\mu_1|^3} = -b \quad \text{when} \quad x = x_e, y = 0.
\]

**Column 3** -
\[
\frac{\partial v_y}{\partial v_x}(x_e,0,0,0) = -2
\]

**Column 4** -
\[
\frac{\partial v_y}{\partial v_y}(x_e,0,0,0) = 0
\]

**Step 2**

Since the values of $\mu, x_e$ are in no way round numbers, this should just be done in Matlab. Using our results from above, we can add the following to the program used to calculate the position of the equilibrium point:

```matlab
% Check Stability of the Equilibrium point L1 by linearization
a=1+2*mu1/(abs(xe+mu2))^3+2*mu2/(abs(xe-mu1))^3
b=-1+mu1/(abs(xe+mu2))^3+mu2/(abs(xe-mu1))^3
A=[0 0 1 0 ; 0 0 0 1; a 0 0 2 ; 0 -b -2 0]
eig(A)
```
Which outputs:

\[ A = \]
\[
\begin{array}{cccc}
0 & 0 & 1.0000 & 0 \\
0 & 0 & 0 & 1.0000 \\
9.2508 & 0 & 0 & 2.0000 \\
0 & -3.1254 & -2.0000 & 0 \\
\end{array}
\]

\[ \text{ans} = \]
\[
\begin{array}{c}
-2.5581 \\
2.5581 \\
-0.0000 + 2.1020i \\
-0.0000 - 2.1020i \\
\end{array}
\]

One eigenvalue is positive, one negative, and two imaginary. The existence of a positive eigenvalue means the equilibrium point is unstable. A practical result of this is that missions to this point require control, called “station keeping” to remain on target.
Problem 5: Nonlinear Stability, Using Manifolds

Let us return to the nonlinear system. We'll see if stable/unstable subspaces and manifolds can be used for fuel efficient missions.

Part 1: Is L1 stable or unstable in the full PCR3BP with the nonlinear equations? Is this true for every direction?

Part 2: Are the conditions of the Hartman-Grobman Theorem satisfied? Why or why not?

Part 3: Assume that the results of the Hartman-Grobman Theorem hold for L1, L2, and L3 regardless of the answer to Part 2 above. Consider the Sun/Earth system. You are working on a mission to place the James Webb Space telescope at the L2 point, which would reduce interference from the Sun by placing the telescope in Earth's shadow. If all the invariant manifolds of L2 pass near the Earth, what types of points may be targeted for a fuel efficient mission? If one of these target states are reached, how many planned maneuvers (aside from corrections) are needed to reach L2?
Solutions

Part 1:

It is unstable because only a single unstable direction in the linearized dynamics is needed to guarantee the equilibrium point is unstable in the nonlinear model.

Part 2:

The conditions are not met. The Hartman-Grobman theorem only applies to hyperbolic equilibrium points (meaning no purely imaginary eigenvalues) and we calculated two imaginary eigenvalues which are complex conjugates of each other.

Part 3:

It turns out that even though the (sufficient) conditions for Hartman-Grobman do not hold, the results still hold in the case of the libration points in the PCR3BP. Thus there is 1-dimensional stable and unstable manifolds which are tangent to the stable and unstable subspaces of the linearized model. (Recall that these subspaces are spanned by the eigenvectors associated with the negative and positive eigenvalues.)

To calculate the stable and unstable manifolds (or at least a quick approximation), calculate the eigenvectors \( v_s, v_u \) associated with the negative/stable and positive/unstable eigenvalues. Let your initial conditions be \( x_{L2} \pm \epsilon v \) where \( x_{L2} \) is the state of the equilibrium point and \( \epsilon \) is a very small positive value. For unstable manifolds which diverge from the equilibrium point integrate forward in time. For stable manifolds, integrate backwards in time to find the trajectory that approaches the starting point.

Modifying a PCR3BP program to calculate these trajectories in the Sun/Earth system for L2 (NASA or Wikipedia work for finding parameters) could look like . . .

```matlab
%xe is the state of L2
a=1+2*mu1/(abs(xe+mu2))^3+2*mu2/(abs(xe-mu1))^3;
b=-1+mu1/(abs(xe+mu2))^3+mu2/(abs(xe-mu1))^3;
A=[0 0 0 1; 0 0 2; a 0 0 2; 0 -b -2 0]; %Linearized Dynamics
[V,D]=eig(A);

stable_direction=V(:,1)';
unstable_direction=V(:,2)';
```
x0 = x0 -.00001*stable_direction; %Initial condition near manifold
time_span=[4 0]; %Time span reversed for stable manifold

options=odeset('RelTol',1e-11,'AbsTol',1e-9);
[t, x] = ode45(@PCR3BP_Equations, time_span, x0, options);

Which gives the following trajectory. Points on the stable manifold approach the equilibrium asymptotically. So, if this manifold passes earth and a satellite may be launched in such a way that its position and velocity places it on this manifold, it will approach L2 with no more fuel needed aside from small corrections.

To investigate further:

It turns out that a generalization of this method is commonly used in mission design to create “low energy transfers”. Trajectories which orbit the libration points have the same type of stability/instability characteristics as the libration point itself. These orbits have stable and unstable manifolds themselves. Calculating these periodic orbits is beyond the scope of this class, but the method is the same.

The successful Genesis mission targeted the stable manifold of an orbit around L1, followed this to spend many months very near the orbit to collect solar wind, until the small perturbation from not being exactly on the manifold pushed it back to earth near the unstable manifold. In the image below, the “outward leg” places the satellite on the stable manifold of an orbit which it approaches asymptotically. In red, it slowly approaches the orbit, shadowing it for 5 revolutions before following the unstable manifold off the orbit and back to Earth. After placing the satellite in the desired state very near the stable manifold, no maneuvers aside from
small corrections were required to complete this complicated mission.

Low Energy Transfers have been studied for missions such as Genesis, Artemis, Dawn, Jupiter Icy Moon Orbiter, and the rescue of Hiten. The concepts studied in class such as linearization, numerically integrating, stability analysis, and periodic behavior are crucial to achieve real mission success.
Problem 6: An Integral of Motion and Forbidden Realms

Part 1: The PCR3BP has an integral of motion. Try to find and verify it. Hint: Even though we are in a rotating frame, it is still called the 'Energy'

Part 2: Since energy is conserved in this system, without thrust the spacecraft will always move with constant energy. Since this is the case, how many coordinates are needed to exactly define the state of a coasting spacecraft? What kind of “space” within ℝ⁴ does the spacecraft move within?

Part 3: For μ = .3, graph some contours of the effective potential between values of -2 and -1.5 (a normal range of energies for many missions). Try to spot where the 5 Lagrange points must be since force is a derivative of the potential.

Part 4: If a region of the xy-plane has a higher effective potential than the spacecraft energy, the spacecraft will not be able to enter these forbidden regions while coasting. Given the following fact:

Energy(L1) < Energy(L2) < Energy(L3) < Energy(L4) = Energy(L5)

in what range of energies did the comet Oterma (see the trajectory in the section introduction) lie?

Solution

Part 1:

\[ \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U(x, y) \] looks a lot like a standard energy formula, and so we calculate its time derivative along a trajectory / Lie Derivative:

\[
\frac{d}{dt} \left( \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U(x, y) \right) \\
= \dot{x}\ddot{x} + \dot{y}\ddot{y} + \frac{dU}{dx}\dot{x} + \frac{dU}{dy}\dot{y} \\
= \dot{x} \left( 2\dot{y} - \frac{dU}{dx} \right) + \dot{y} \left( -2\dot{x} - \frac{dU}{dy} \right) + \frac{dU}{dx}\dot{x} + \frac{dU}{dy}\dot{y}
\]
\[ 2 \dot{x} \dot{y} - \frac{d \bar{U}}{dx} \dot{x} + -2 \dot{x} \dot{y} - \frac{d \bar{U}}{dy} \dot{y} + \frac{d \bar{U}}{dx} x + \frac{d \bar{U}}{dy} y = 0 \]

and so \( E(x, y, \dot{x}, \dot{y}) = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + \bar{U}(x, y) \) is an integral of motion. It should be noted that some people in the celestial mechanics community use \( J = -2E \), the Jacobi Constant. These are just scalar multiples of one another and so either is acceptable.

**Part 2:** Looking at \( E(x, y, \dot{x}, \dot{y}) = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + \bar{U}(x, y) = \text{constant} \), we can see that once we know the energy of the spacecraft we can determine the value of a fourth variable by plugging three known variable in to the energy formula and solving for the unknown. In this sense, along with other systems with integrals of motion, the satellite is said to move on a \( 4-1=3 \)-dimensional surface/manifold within phase space. It is this reduction of dimension that gives the invariant manifolds of periodic orbits in the PCR3BP some of their amazing properties (See the reference by Koon, et al.)

You might have noticed that you are probably only able to solve for \( \pm \) the desired fourth coordinate. There might be two distinct points that solve the equation, however by the continuity and smoothness of the motion the spacecraft will not teleport between points and so will stay on its original surface.

**Part 3:**
Those festive renditions of the effective potential are given by:

\[ \mu = 0.3 \]

```matlab
figure
[X,Y]=meshgrid(-1.7:0.05:1.7);
r1=((X - (-mu)).^2+Y.^2).^(1/2);
r2=((X - (1-mu)).^2+Y.^2).^(1/2);
Z=[(-1/2)*(X.^2 + Y.^2)-(1-mu)./r1-mu./r2];
contour3(X,Y,Z,-2:.001:-1.38)
axis([-1.7 1.7 -1.7 1.7 -2 -1.38])
```

The Lagrange points would have to lie at either a maximum or saddle point of the potential in order for the resulting 'force' of the potential to be zero. The holes in the plot are the huge potential wells for the primary and secondary bodies.

**Part 4:** Considering the shape of the effective potential and the inequalities of the Lagrange point energies, we have five cases of areas that are forbidden to move to:

![Diagram](attachment:diagram.png)

Oterma appears to only be able to move to and from the region near the Sun only by passing by close to Jupiter. This suggests the energy is between that of L2 and L3, which is indeed the case.
Continuing Example: Nonlinear ODE with parameter bifurcation

Problem 1: Classification, First Order Form, Equilibria

Consider the system defined by the equation \( \ddot{x} + \mu x + x^2 = 0 \) where \( \mu \) can be any real number.

Part 1: What type of system is this? Discuss things like order, linearity, etc.

Part 2: Put the system in first order form.

Part 3: Are there any equilibrium points? If so, how many and what are they? Does anything change when \( \mu \) is positive, negative, or zero?

Solution

Part 1:

This is a second order (\( \ddot{x} \)), nonlinear (\( x^2 \)), autonomous, homogeneous ODE.

Part 2:

Let \( x_1 = x, x_2 = \dot{x} \). We can then convert the system into the first order form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-\mu x_1 - x_1^2
\end{bmatrix}
\]

Part 3:

To find equilibria we set the right side equal to zero. This gives us \( x_2 = 0, -\mu x_1 - x_1^2 = -x_1(\mu + x_1) = 0 \). Thus we have equilibria at \((0,0)\) and \((-\mu, 0)\) if \( \mu \neq 0 \) and just a single equilibrium point at the origin otherwise. Note that depending on the sign of \( \mu \) one of the equilibrium points can be to the left or the right of the origin.
**Problem 2: Linearization and Lyapunov's Indirect Method**

**Part 1:** Using the first order form and the equilibrium points you calculated before, linearize the system at its equilibrium points.

**Part 2:** Are the linearizations themselves stable or unstable for each equilibrium point? Does anything change when $\mu$ is positive, negative, or zero?

**Part 3:** Can we say whether the equilibrium points are stable or unstable in the full nonlinear case?

**Solution**

**Part 1:**

From before, we had that the first order form was $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\mu x_1 - x_1^2 \end{bmatrix} = f(x)$ with equilibria $(0,0)$ and $(-\mu, 0)$. Let $(x_{1e}, x_{2e})$ be coordinates for some equilibrium point and $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{1e} \\ x_2 - x_{2e} \end{bmatrix}$. Evaluating the Jacobian matrix of $f$ at an equilibrium point gives us the linearized model near this point as $\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\mu - 2x_{1e} & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$.

Thus for our two equilibria we have:

At $(0,0)$, $\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\mu & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$. At $(-\mu, 0)$, $\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$.

**Part 2:**

We will now analyze the stability of the linearized models by calculating the eigenvalues of the state transition matrices. Begin by assuming $\mu \neq 0$.

**Case 1:** Equilibrium at $(0,0), \mu > 0$

The characteristic polynomial is $det \begin{bmatrix} \lambda & -1 \\ \mu & \lambda \end{bmatrix} = \lambda^2 + \mu = 0$ and so the eigenvalues are $\lambda = \pm \sqrt{-\mu}$. Since $\mu > 0$ the eigenvalues are purely imaginary so this linearized
model is stable (in the sense of Lyapunov), however it is not asymptotically stable.

**Case 2:** Equilibrium at \((0,0), \mu<0\)

The characteristic polynomial is 
\[
\det \begin{bmatrix} \lambda & -1 \\ \mu & \lambda \end{bmatrix} = \lambda^2 + \mu = 0
\]
and so the eigenvalues are \(\lambda = \pm \sqrt{-\mu}\). Since \(\mu<0\) one of the eigenvalues is a positive real number, this linearized model is unstable.

**Case 3:** Equilibrium at \((-\mu,0), \mu>0\)

The characteristic polynomial is 
\[
\det \begin{bmatrix} \lambda & -1 \\ -\mu & \lambda \end{bmatrix} = \lambda^2 - \mu = 0
\]
and so the eigenvalues are \(\lambda = \pm \sqrt{\mu}\). Since \(\mu>0\), one of the eigenvalues is a positive real number, this linearized model is unstable.

**Case 4:** Equilibrium at \((-\mu,0), \mu<0\)

The characteristic polynomial is 
\[
\det \begin{bmatrix} \lambda & -1 \\ -\mu & \lambda \end{bmatrix} = \lambda^2 - \mu = 0
\]
and so the eigenvalues are \(\lambda = \pm \sqrt{\mu}\). Since \(\mu<0\) the eigenvalues are purely imaginary so this linearized model is stable (in the sense of Lyapunov), however it is not asymptotically stable.

If \(\mu=0\) then there is only one equilibrium point and the characteristic polynomial is \(\lambda^2 = 0\) and so there is only one eigenvalue, 0. This is the case of a repeated eigenvalue with zero real part, which is unstable (diverges linearly in time rather than exponentially).

**Part 3:** The unstable equilibria above in the \(\mu \neq 0\) cases all had an eigenvalue with a positive real part. As a result we may conclude that these are unstable in the linear case as well. For the stable (in the sense of Lyapunov) equilibria, there were (exclusively, in this case) eigenvalues with zero real part, so we cannot conclude either way if these will be stable in the nonlinear case. Similarly, we can draw no conclusions about the single equilibrium point when \(\mu = 0\).
**Problem 3: Integrals of Motion**

**Part 1:** Try to find an integral of motion for the system. Explain your reasoning and verify it is an integral of motion using Lie derivatives.

**Part 2:** Use the formula for your integral of motion to graph a potential function. Nothing too precise, try to just get the general shape but be sure to put the equilibrium values in the right places. Make a graph for the three cases of $µ$.

**Solution**

**Part 1:**

Although there are certainly other types of integrals of motion, looking for something akin to energy is usually a good idea. This is especially true for systems with a simple $\dot{x}_1=x_2$ that looks equivalent to “the derivative of position is velocity” and where the other terms depend on “position” only. Start with the usual rule:

$$\text{Total Energy} = \text{Kinetic Energy} + \text{Potential Energy}$$

$\frac{1}{2}x_2^2$ looks like an analogue of kinetic energy. In a system where energy is conserved we have potential forces only so if $U$ is the potential energy, $F=-\frac{dU}{dx}$. Following this example we want a function $U$ such that $\dot{x}_2=-\frac{dU}{dx_1}$. This would be satisfied by $µ\frac{x_1^2}{2}+\frac{x_3^3}{3}$. Let us try the following guess:

$$E=\frac{1}{2}x_2^2+µ\frac{x_1^2}{2}+\frac{x_3^3}{3}$$

Let's verify this is an integral of motion by verifying that $\dot{E}=L_fE=0$.

$$L_fE=\left[µx_1+x_2^2\right]x_2\left[\frac{x_2}{-µx_1-x_2}\right]=\left(µx_1+x_2^2\right)x_2+x_2\left(-µx_1-x_2\right)=0$$

so indeed $E$ is an integral of motion.
Part 2: Here is a plot of the potential function $U$ for three values of $\mu$. 
Problem 4: Lyapunov Functions, Lyapunov's Direct Method

Part 1: Try to find a Lyapunov function for the equilibria for which stability has yet to be determined. Does our integral of motion work? Do we need to make any changes to it? Does anything change when $\mu$ is positive, negative, or zero?

Part 2: Combine the results with those of the earlier linearization to describe the stability of all the equilibria for the different values of $\mu$.

Solution

If an integral of motion is found for a system, it is often a good starting point for showing an equilibrium point is stable using the Lyapunov Function method. However, since the time derivative is always equal to zero, we will not be able to prove asymptotic stability doing this (why?) even if it were possible with another function. The reason for this is that we always have $\dot{V} = \dot{E} = L_f E = 0$ and so $\dot{V}$ is negative semidefinite. This only leaves us the work to check if the integral of motion satisfies the constraints on $V$ itself.

Our plot of the potential function $U$ will be helpful here for intuition. Note that $E(x_1,x_2) \geq U(x_1)$ and $E(x_1,x_2) > U(x_1)$ if $x_2 \neq 0$.

Case 1: $\mu > 0$

In this case the equilibrium point we are analyzing is the origin. $E(0,0) = 0$. However $E$ is not positive over the entire domain so the equilibrium point cannot be shown to be globally stable. But from looking at our potential function, we see that we can easily restrict the domain to get local stability.

Start with the fact $E(x_1,x_2) \geq U(x_1) = \mu \frac{x_1^2}{2} + \frac{x_1^3}{3}$. Let’s restrict the domain to $x_1 \geq -\mu$. We will look for points where $E = 0$. This implies $\mu \frac{x_1^2}{2} = -\frac{x_1^3}{3}$. As we know, 0 is a solution, however for nonzero values we can cancel terms to get $\mu = -\frac{x_1}{3}$ and so $-\frac{3}{2} \mu = x_1$. But due to our domain restriction $-\frac{3}{2} \mu < -\mu < x_1$ and so there is no solution other than 0. Thus in our
restricted domain \( E(x_1, x_2) = \frac{x_2^2}{2} + U(x_1) > 0 + 0 = 0 \) for \( (x_1, x_2) \neq (0,0) \).

So \( E \) restricted to the domain above is a function that satisfies the requirements for a (local) Lyapunov function:

1. \( E(0,0) = 0 \).
2. \( E(x_1, x_2) > 0 \) for \( (x_1, x_2) \neq (0,0) \).
3. \( \dot{E} = 0 \)

and so the origin in a locally stable equilibrium point in this case.

**Case 2:** \( \mu < 0 \)

This is very similar to the above, but with two important modifications. The first is to correct the problem that \( E(-\mu, 0) < 0 \). This is easily corrected by considering the function

\[ V(x_1, x_2) = E(x_1, x_2) - E(-\mu, 0) \]

which equals zero at the equilibrium point and also \( \dot{V} = \dot{E} \).

Another change is the restriction on the domain. In this case \( x_1 > 0 \) will do the trick. With these two modifications, local stability can be shown as it was for Case 1.

**Case 3:** \( \mu = 0 \)

Using \( E \) as a Lyapunov Function candidate will clearly not work in this case since there is no neighborhood where \( E(0,0) = 0 \) and \( E(x_1, 0) \geq 0 \) for all \( x_1 \) in the neighborhood. This can be seen because \( E(x_1, 0) = 0 \frac{x_1^2}{2} + \frac{x_1^3}{3} < 0 \) for any negative \( x_1 \).

In fact, we can see that in this case the single equilibrium point is unstable. With \( \mu = 0 \) the system reduces to

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
x_2 \\
-x_1
\end{bmatrix}
\]

Let \( x_0 = (-\epsilon, 0) \) for an arbitrarily small \( \epsilon > 0 \). \( x_2 \) will immediately become and stay negative for all time, and thus \( x_1 \) will become increasingly negative for all time. This violates the definition of stability for an equilibrium point, and so the origin is unstable in this case.
Problem 5: Phase Portrait

- Draw phase portraits for the three cases of $\mu$. Place the equilibria and use the stability results to show how nearby points will behave. Use the potential functions to help fill in the rest. In particular, think about how a marble starting atop the unstable equilibrium would move with a slight positive, negative, or zero velocity.
- What happens to the general behavior of the origin as $\mu$ passes from negative to positive?

Discussion

Use pplane to make a phase portrait for the three cases of $\mu$. Does it match up well with your drawing? Note the change in stability at the origin and the other equilibrium as $\mu$ passes from negative to positive. At $\mu=0$ the single equilibrium point is nonhyperbolic and forms a cusp.