Problem 1 (25pts): Just a review...

Consider the following ordinary differential equation (ODE):
\[
\frac{dx}{dt} = \begin{cases} 
  x - 4 & \text{if } t < 5 \\
  2 - x & \text{if } t \geq 5 
\end{cases}
\]

1. What type of ODE is it?
2. Find a solution of this equation satisfying \( x(0) = 4 \). Describe the qualitative behavior of this solution.
3. Find a solution of this equation satisfying \( x(0) = 3 \). Describe the qualitative behavior of this solution.
4. Describe the qualitative behavior of any solution of this system as \( t \to \infty \).

Problem 2 (25pts): On the notion of state

For each of the following systems, describe the state of the system and write the equations as a first order system, if possible.

Note that \( x', x'',... , x^{(n)} \) are used to denote the first, second and \( n^{th} \) derivative of the function \( x(t) \) with respect to the independent variable \( t \). All the variable are assumed to be real. The letters \( x,y,z \) denote variables (unknown functions of \( t \)).

1. \( x''' = x'' \)
2. \( (x')^3 + \sin(t).x^2 = 2 \)
3. \( \sin(x^{(3)}) + x = \ln(t.x') \)
4. \( x''.x' + (x''' + x)^2 = \sin(x'') \)
5. \( x''y'' + x.y'.z'' = 2 \)
6. \( x.y'' + (x'.y')^2 = -5 \); \( x^{(4)}.y = \frac{x}{y} \)

Problem 3 (25pts): On the notion of flow

1. Determine the flow defined by of the following differential equation \( \frac{dx}{dt} = x^2 \). What is the domain of definition of the flow?
2. Let us now consider the following system \( \frac{dx}{dt} = x^{1/3} \). Does this define a flow? Explain.
Problem 4 (25pts): Phase portraits

Determine the equation of the phase trajectories for the given systems and sketch several representative trajectories. Use arrows to indicate the direction of movement along those trajectories.

1. \[ \frac{dx}{dt} = y; \quad \frac{dy}{dt} = -x \]
2. \[ \frac{dx}{dt} = x \cdot y; \quad \frac{dy}{dt} = x^2 \]
3. \[ \frac{dx}{dt} = y^2; \quad \frac{dy}{dt} = -x \cdot y \]

(all the variables are real, 1-dimensional variables).

Suggestions for further study

You should be able to solve linear first order ODEs in 1-dimension, and have heard about the notion of separation of variables. If you haven’t seen that during your undergraduate studies, please read chapter 2 of Greenberg’s book, for example. Doing a few exercise of this chapter of the first one would also be good if you don’t have any background on ODEs.