MAE 200A: Engineering Analysis I – Midterm #1

This is a take-home exam

Assigned Thursday, October 25th, 2012.
Due, beginning of the lecture, on Tuesday October 30th, 2012.

Guidelines

• Please use a “blue book” to write your answers. Clear explanations of your reasoning and a clean copy are required to receive full credit for a question. In particular, prove all of your answers (or disprove the statements by counter-examples, for example).

• The exam should be done individually. No discussion allowed (except with textbooks and yourself!)

Problem 1 (10pts)

For the following system, describe the solution set, both geometrically and algebraically.

\[
\begin{align*}
(1) & \begin{cases} 
    x_1 - x_2 = 1 \\
    x_1 - 2x_2 = 2 \\
    2x_1 - x_2 = 4 
\end{cases} & (2) & \begin{cases} 
    x_1 + 2x_2 + 3x_3 = 4 \\
    4x_1 + 3x_2 + 2x_3 = 1 
\end{cases}
\end{align*}
\]

Problem 2 (15pts)

1. Give an example of subsets of \( \mathbb{R}^3 \) that is:
   (a) closed under addition, but not closed under scalar multiplication.
   (b) closed under scalar multiplication, but not closed under addition.

2. Define the “Spam” of a subset \( S \) of a vector space \( V \) as:
   \[ \text{Spam}(S) = \{ a.u + b.v \mid a, b \in \mathbb{R} \text{ and } u, v \in S \} \]
   Does \( \text{Spam}(S) = \text{Span}(S) \)?

Problem 3 (20pts)

1. Let us consider \( \mathbb{R}^n \) equipped with two bases \( e = \{ e_1, \ldots, e_n \} \) and \( f = \{ f_1, \ldots, f_n \} \) related by: \( f_j = \sum_{j=1}^n a_{ij} e_j \); \( j = 1, \ldots, n \) we define the change of basis matrix \( A = (a_{ij}) \). We assume that \( A^T.A = A.A^T = I \), the identity matrix. Matrices, \( M \), satisfying such a relation, \( M^T.M = M.M^T = I \), are called orthogonal. Show that \( A \) is invertible and compute the inverse of \( A \). What is the relation between the coordinates \( (x_1, \ldots, x_n) \) of a vector in the \( e \)-basis and the coordinates \( (y_1, \ldots, y_n) \) of the same vector in the \( f \)-basis?

2. For an arbitrary (real) square matrix \( B \), one can show that there exist \( U \) and \( V \) (real) orthogonal matrices, and \( D \) a diagonal matrix such that \( B = U^T.D.V \). This is the singular value decomposition of \( B \) and the (non-zero) diagonal elements of \( D \) are the singular values of \( B \). Interpret the singular value decomposition in terms of change of basis and draw a commutative diagram.

3. Do the eigen-values of a square matrix always equal its singular values? If so, prove it. If not, show a counter-example.
Problem 4 (15pts)

This problem looks at the modeling the simplest model for a spacecraft rotational dynamics. We assume that the spacecraft is equipped with small thrusters orthogonal to its major axis, which allows it to operate impulsive changes in its angular velocity, $\omega$, along that axis. Between two impulsive maneuvers, the angular velocity is constant and the orientation of the spacecraft (relative to a fixed direction in space) is determined by an angle $\theta(t) = \omega t + \theta_0$. Let's assume that the spacecraft starts with zero angular velocity for $T < t_0$ and that $\theta(t) = 0$ on this interval of time. The spacecraft then performs a sequence of 4 impulsive maneuvers at times $t_0 < t_1 < t_2 < t_3$. In each interval $[t_i, t_{i+1}]$ the angular velocity is denoted by $\omega_i$.

- Assuming there is no limitation to the values of the $\omega_i$, describe the possible set of functions $\theta(t)$ on the interval $[t_0, t_3]$. Show in particular, that this set is a vector space, $V$.

- Describe a basis for $V$ and determine the dimension of $V$.

- Consider the mapping $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(\omega_0, \omega_1, \omega_2) = (\theta(t_1), \theta(t_2), \theta(t_3))$. Is this mapping linear? What is its range. Is there a unique way to achieve given orientations at these given times? Justify.

Problem 5 (20pts)

To a first approximation, the Earth can be considered as a perfect sphere rotating uniformly relative to inertial reference frames centered at the center of mass of the Earth. If $\vec{r}$ denotes the position vector of a point fixed relative to the surface of the Earth, then classical mechanics, indicates that the inertial velocity and acceleration of that point is given as $\vec{v} = \vec{\omega} \times \vec{r}$ and $\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$, respectively. Here $\vec{\omega}$ denotes the angular velocity of the Earth relative to inertial space. Let us thus consider the mappings $V$ and $A$ which associate the inertial velocity and acceleration to every point $\vec{r}$, according to the above laws.

1. Show that $V$ and $A$ are linear operators.

2. Show that both $V$ and $A$ have zero as eigen-value and find a corresponding eigen-vector. What are the geometric and algebraic multiplicities of this eigenvalue?

3. Describe the canonical forms of these operators and find a basis in which both $V$ and $A$ are in such a form.

Problem 6 (20pts)

In this problem we consider a 3-link robot arm as illustrated in Figure 1. The coordinates of the end-tip of the arm (in the standard basis of $\mathbb{R}^2$) is related to the link lengths and angles as follows:

$$
\begin{align*}
x &= l_1. \cos \theta_1 + l_2. \cos(\theta_1 + \theta_2) + l_3. \cos(\theta_1 + \theta_2 + \theta_3) \\
y &= l_1. \sin \theta_1 + l_2. \sin(\theta_1 + \theta_2) + l_3. \sin(\theta_1 + \theta_2 + \theta_3)
\end{align*}
$$

1. Assuming the link lengths, $l_i$, can be chosen to be any value (even negative), for which value of $\theta_1, \theta_2, \theta_3$ can you reach any given $(x, y)$ position? In those cases, how many choices of lengths $(l_1, l_2, l_3)$ do you have to achieve a desired position? If multiple choices exists, can you always choose a choice with positive link length?

2. For the value of $\theta_1, \theta_2, \theta_3$ where you cannot reach any $(x, y)$ position by controlling the link lengths, describe the set of positions $(x, y)$ that can be reached. Express you answer in terms of span of particular vectors.

Figure 1: A planar 3-link robot arm.