In the following, \((\mathbb{U}, <\cdot, \cdot>_{\mathbb{U}})\) and \((\mathbb{V}, <\cdot, \cdot>_{\mathbb{V}})\) represent finite dimensional inner-product spaces over the field of real or complex numbers, denoted generically by \(\mathbb{K}\). The overbar denotes complex conjugation. For extensions to the infinite dimensional cases, see reference by N. Young.

**Projections**

**Definition: Projection operator.**

1. An operator \(P : \mathbb{U} \to \mathbb{U}\) such that \(P \circ P = P\) is called a projection. In particular, \(\text{Null}(P) \oplus \text{Range}(P) = \mathbb{U}\) and \(P\) restricted on \(\text{Range}(P)\) is the identity operator.

2. A projection is called orthogonal if \(\text{Null}(P)\) is orthogonal to \(\text{Range}(P)\).

**Orthogonal transformations**

**Definition: Inner-product and norm preserving transformation**

1. A linear operator \(A : \mathbb{U} \to \mathbb{V}\) is an inner-product preserving transformation if for all \(x, y \in \mathbb{U}\), \(<A x, A y>_{\mathbb{V}} = <x, y>_{\mathbb{U}}\).

2. A linear operator \(A : \mathbb{U} \to \mathbb{V}\) is a norm preserving transformation if for all \(x \in \mathbb{U}\), \(||A x||_{\mathbb{V}} = ||x||_{\mathbb{U}}\).

**Lemma:**

1. An inner-product preserving transformation is norm preserving. Reciprocally, if the norm derives from an inner product, then a norm preserving transformation is an inner-product preserving transformation.

2. An inner-product or norm preserving transformation is always injective.

**Definition: Orthogonal transformations**

1. A norm (or inner-product) preserving transformation \(A : \mathbb{U} \to \mathbb{U}\) is called an orthogonal transformation or an isometry.

2. The set of orthogonal transformations on \(\mathbb{U}\) is denoted \(\mathcal{O}(\mathbb{U})\). The set of orthogonal transformation on \(\mathbb{R}^n\) is denoted \(\mathcal{O}(n)\) and is referred to as the orthogonal group.
Theorem:
1. For any $A, B \in O(n)$, $A.B \in O(n)$, $A^{-1}$ exist and is also orthogonal.

2. The matrix representation of an orthogonal transformation satisfy the relation: $A^T.A = I$. In particular, $A^{-1} = A^T$.

Definition: Rotations (special orthogonal group)
An orthogonal transformation satisfying $det(A) = 1$ is called a rotation. The set of rotation is denoted $SO(n)$ and called the special orthogonal group. The previous theorem apply to the elements of $SO(n)$ as well.

Adjoint and Self-Adjoint

Theorem/definition: Adjoint operators
For any operator $A : U \rightarrow V$, there exists a unique operator $A^* : V \rightarrow U$ such that, for all $x \in U$ and $y \in V$,

$$< A.x, y >_V = < x, A^*.y >_U$$

The operator $A^*$ is called the adjoint of $A$.

Lemma:
1. If $A$ is the matrix representation of a linear operator in two given basis, $(e)$ and $(f)$, then $A^T$ is the matrix representation of the adjoin operator in the same bases.

2. The following relations hold:
   $(A^*)^* = A$ ; $(B.A)^* = A^*.B^*$ ; $(\lambda.A + \mu.B)^* = \lambda.A^* + \mu.B^*$.

Theorem: The range and null spaces of an operator $A$ and its adjoint, satisfy the following relations

$$Range(A) \perp Null(A^*) \text{ and } Null(A) \perp Range(A^*)$$

Definition: Self-Adjoint operator
A linear operator $A : U \rightarrow U$ is called self-adjoint (or Hermitian) if $A = A^*$.

Spectral Theorem for Self-Adjoint operators

If $A : U \rightarrow U$ is self-adjoint then the following hold:

1. The eigen-values of $A$ are real.

2. $A$ is diagonalizable. That is, there exists a basis of $U$ formed of eigen-vectors of $A$.

3. The eigen-vector basis can be chosen to be orthonormal.