Why is the tropospheric mean lapse rate \( \Delta 6.5 \text{ K km}^{-1} \)?

- Thermodynamic Principles
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### 2. Thermodynamic quantities

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<td>heat energy</td>
<td>J</td>
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<td>q=Q/M</td>
<td>specific heat energy</td>
<td>J kg(^{-1})</td>
</tr>
<tr>
<td>U</td>
<td>internal energy</td>
<td>J</td>
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<tr>
<td>u=U/M</td>
<td>specific internal energy</td>
<td>J kg(^{-1})</td>
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<tr>
<td>W</td>
<td>work energy</td>
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<td>w=W/M</td>
<td>specific work energy</td>
<td>J kg(^{-1})</td>
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<tr>
<td>(c_p)</td>
<td>specific heat capacity (const. p)</td>
<td>J kg(^{-1}) K(^{-1})</td>
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<td>(l_v)</td>
<td>specific latent heat of vaporization</td>
<td>J kg(^{-1})</td>
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3. Thermodynamic principles

- Complete state of pure substance (gas, liquid) uniquely determined by two properties called **state variables**:
  - Properties of system \((p, T, V, \rho)\) not process \((W, Q)\)
  - **Intensive** properties are mass-independent (lowercase, except \(T\))
  - **Specific** properties are intensive, e.g., \(v \equiv V/M, u \equiv U/M\)
- Two types of energy exchange are possible:
  - **Work** \(W\) exchanges mechanical energy, e.g., expansion, compression
  - **Heat** \(Q\) exchanges thermal energy, e.g., conduction, radiation
- Thermodynamic processes are changes between two states
- Systems seek **thermodynamic equilibrium**, a combination of **mechanical equilibrium** and **thermal equilibrium**
4. First Law

**Adiabatic** processes require work and involve no heat transfer \((Q = 0)\)
\[
\Delta u = -w \\
du = -\delta w \tag{1}
\]

**Diabatic** processes involve exchanges of heat \(q\).
\[
\Delta u = q - w \\
du = \delta q - \delta w \tag{2}
\]

The **First Law of Thermodynamics** states that the change in internal energy of a system equals the heat \(Q\) transferred into the system minus the work \(W\) performed by the system.
5. Work, expansion, and compression

Work is done by expanding and on compressing parcels of air . . .

\[ W = p \Delta V \]
\[ \delta W = p \, dV \]  (3)

Inserting this into (2) we obtain

\[ \Delta U = Q - p \Delta V \]
\[ du = \delta q - p \, dv \]  (4)

The troposphere is a heat engine that converts some absorbed solar radiation (heat) into motion (work), and any remaining heat warms the air.
Figure 1: Work done by a gas on piston represented on p-V diagram. Expanding from state P to state Q requires $W = pdV$ (Wallace and Hobbs, 2006, Fig. 3.4)
By definition, the instantaneous system configuration (i.e., state) determines all state variables. An incremental (microscopic) change in a state variable such as pressure or volume is therefore indicated with the perfect differential symbol “d”, e.g., d\(p\), d\(V\).

Processes like mechanical work (\(pdV\)) depend very much on the path taken such that \(\oint dW \neq 0\).

For such processes, incremental changes in the state of the system are indicated with the imperfect differential symbol “\(\delta\)”, e.g., \(\delta W, \delta q\).

Macroscopic changes in a state or energy variable are indicated by the finite change symbol “\(\Delta\)”, e.g., \(\Delta p, \Delta W\).
7. Energy and enthalpy

**Internal Energy** $U$ is a state variable useful for diagnosing isochoric processes (constant volume, $dv = 0$):

$$du = \delta q - p \, dv$$

(5)

**Isochoric** processes transform heat into internal energy.

“internal energy” implies that $u$ does not include energy defined by reference to external coordinate systems, such as potential energy (gsz) and kinetic energy ($v2/2$).

**Enthalpy** $H \equiv U + pV$ is a state variable useful for diagnosing isobaric processes (constant pressure, $dp = 0$):

$$dh = du + p \, dv + v \, dp$$

$$= du + \delta w + v \, dp$$

$$= \delta q + v \, dp$$

(6)

Isobaric processes transform heat into enthalpy.
Figure 2. An isotherm and adiabat on a p-V diagram (Wallace and Hobbs, 2006, Figure 3.5)
8. Heat capacity

Observations show that temperature change at constant pressure or volume is proportional to heat absorbed. Proportionality constant is specific heat capacity

\[ c_p = \left( \frac{\delta q}{dT} \right)_p \]

\[ c_v = \left( \frac{\delta q}{dT} \right)_v \]

(7)

where the subscripts denote isobaric (constant \( p \)) and isochoric (constant \( v \)) processes.

Theoretically, \( c_p = c_p(p, T) \) and \( c_v = c_v(p, T) \) but measurements show that, at \( p, T \) relevant to Earth system

\[ c_p = 1004.7 \text{ J kg}^{-1} \text{ K}^{-1} \text{ (air)}, 4187 \text{ J kg}^{-1} \text{ K}^{-1} \text{ (liquid water)} \]  \hspace{1cm} (8)

\[ c_v = 717.5 \text{ J kg}^{-1} \text{ K}^{-1} \text{ (air)} \] \hspace{1cm} (9)
9. Dry and moist energy and enthalpy

Since \( c_p \), \( c_v \) are constants we may integrate (7)

\[
\begin{align*}
  u &= c_v T \\
  h &= c_p T
\end{align*}
\]

(10)

where constants were chosen so \( U = H = 0 \) J at \( T = 0 \) K.

Other forms of energy may be added to (10). Meteorologists define dry static energy \( D \) and dry static enthalpy \( DH \) by adding potential energy \( \Phi = M g_s z \) to (10)

\[
\begin{align*}
  d &= u + \varphi = c_v T + g_s z \\
  dh &= h + \varphi = c_p T + g_s z
\end{align*}
\]

(11)

moist static energy \( J \) and moist static enthalpy \( JH \) includes potential energy and latent energy \( L = M lq \):

\[
\begin{align*}
  j &= d + lq = c_v T + g_s z + lq \\
  jh &= dh + lq = c_p T + g_s z + lq
\end{align*}
\]

(12)

\( U \), \( H \), \( D \), \( DH \), \( J \), and \( JH \) are all state functions.
Adiabatic processes exchange no heat with the environment. For adiabatic processes, the first law (5,6) reduces to

\[ c_v \, dT + p \, dv = 0 \]
\[ c_p \, dT - v \, dp = 0 \]

Substituting the Ideal Gas Law \( v = RT/p \) into (13)

\[ c_p dT = \frac{RT}{p} \, dp \]

\[ \frac{dT}{T} = \frac{R dp}{c_p p} \]

\[ \frac{T}{T_0} = \left( \frac{p}{p_0} \right)^{R/c_p} \quad (14) \]
11. Conserved properties

Defining \( \kappa \equiv R/c_p \approx 0.286 \), \( \theta \equiv T_0 \), and \( p_0 \equiv 1000 \text{ mb} \)

\[
\frac{\theta}{T} = \left( \frac{p_0}{p} \right)^\kappa \quad \text{or} \quad \theta = T \left( \frac{p_0}{p} \right)^\kappa \tag{15}
\]

Potential temperature \( \theta \) [K] is the temperature of air compressed or expanded adiabatically to \( p_0 \equiv 1000 \text{ mb} \).

Adiabatic vertical motion of an air parcel causes \( T \) to change due to expansion work performed. However, Poisson’s equation shows that potential temperature \( \theta \) is conserved under adiabatic transformations.

Similarly, dry static enthalpy \( DH \) (11) is conserved for dry air processes and moist static enthalpy \( JH \) (12) is conserved for pseudo-adiabatic moist air processes.
12. Dry lapse rate

The environmental lapse rate $\Gamma$ quantifies the local rate of change of temperature

$$\Gamma(z) \equiv -\frac{dT}{dz} \quad (16)$$

The mean tropospheric lapse rate $\Gamma \approx 6.5 \text{ K km}^{-1}$, why?

Consider a dry air parcel moving vertically in a hydrostatic atmosphere. The parcel always seeks mechanical equilibrium with the environment. Using $dp = -\rho g_s \, dz$ in (13),

$$c_{p,d} \, dT + v \rho g_s \, dz = 0$$

$$- \frac{dT}{dz} = \frac{g_s}{c_{p,d}} \equiv \Gamma_d$$

where $\Gamma_d \approx 9.8 \text{ K km}^{-1}$ is the dry adiabatic lapse rate (DALR).

Parcels ascend/descend until $(T/p, \text{ or } \rho)_{\text{parcel}} = (T/p, \text{ or } \rho)_{\text{environment}}$

ie. the level of neutral buoyancy.

Problem: $\Gamma_d$ is about 50% larger than the observed $\Gamma$. 
13. Moist adiabatic lapse rate

The moist adiabatic lapse rate (MALR) derives from the same principles as the DALR (Tsonis, 2002, p. 104). With the gas constants corrected for the presence of water vapor,

\[ \Gamma_m = \frac{g_s}{c_p} = \frac{g_s}{c_{p,d}(1 + 0.87r)} \]

\[ \Gamma_m = \frac{\Gamma_d}{(1 + 0.87r)} = \Gamma_d(1 - 0.87r) \]

On Earth \( \Gamma_m \approx \Gamma_d \) and \( 9.5 \leq \Gamma_m < 9.8 \text{ K km}^{-1} \).

So, unsaturated water vapor cannot explain the discrepancy between dry and observed lapse rates.
14. Phase changes

Chemicals may be in thermodynamic equilibrium with more than one phase – heterogeneous systems. For isobaric transformations of heterogeneous systems, $dT = 0$. Heat transfer converts mass from one phase to another and work changes the volume of the system.

Specific latent heat of transformation $l$ is heat absorbed during phase transformation $l \equiv \delta q_p = dh \quad (18)$

- Evaporation: Liquid $\rightarrow$ Gas, $l_v(0^\circ C) = 2.5104 \times 10^6 \text{ J kg}^{-1}$
- Condensation: Gas $\rightarrow$ Liquid
- Fusion: Liquid $\rightarrow$ Solid, $l_f(0^\circ C) = 0.334 \times 10^6 \text{ J kg}^{-1}$
- Melting: Solid $\rightarrow$ Liquid
- Deposition: Gas $\rightarrow$ Solid, $l_s(0^\circ C) = 2.844 \times 10^6 \text{ J kg}^{-1}$
- Sublimation: Solid $\rightarrow$ Gas
Figure 3. (a) unsaturated and (b) saturated air over a liquid water surface. Arrows indicate condensation/evaporation rates (Wallace and Hobbs, 2006, Figure 3.8).
15. Clausius-Clayperon equation

This expression relates equilibrium vapor pressure \( \tilde{\varepsilon} \) in a heterogeneous system at temperature \( T \) to phase changes \( \Delta v \):

\[
\frac{d\tilde{\varepsilon}}{dT} = \frac{l}{T \Delta v}
\]

where \( \Delta v \equiv v_{\text{initial}} - v_{\text{final}} \) (19)

For condensed/gas transformation \( v_{\text{gas}} >> v_{\text{condensed}} \), so

\[
\Delta v \approx \frac{R_v T}{\tilde{\varepsilon}}
\]

(20)

and

\[
\frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}} = \frac{l \ dT}{R_v T^2}
\]

(21)
16. Saturation

l is approximately constant at Earth system T. Thus (21) may be integrated:

\[
\tilde{e}(T) = e_0 \exp \left[ \frac{l}{R_v} \left( \frac{1}{T_0} - \frac{1}{T} \right) \right]
\]

use \( T_0 = 273.15 \) and \( \tilde{e}_0 = 6.11 \) mb to parameterize \( e \) (mb) and T(K):

\[
\log_{10} \tilde{e}_i(T) \approx 9.401 - \frac{2354}{T}
\]

(22)

\[
\tilde{e}_i(T) \approx 6.11 \exp \left( 19.83 - \frac{5417}{T} \right)
\]

(23)

\[
\log_{10} \tilde{e}_i(T) \approx 10.56 - \frac{2667}{T}
\]

\[
\tilde{e}_i(T) \approx 6.11 \exp \left( 22.49 - \frac{6142}{T} \right)
\]

(24)

moist air is often characterized by its properties relative to saturation:

ie. Relative Humidity = RH = \( \frac{e}{\tilde{e}} \)
Figure 4. Saturation vapor pressure $\tilde{e}_l$ over water and $\tilde{e}_l - \tilde{e}_i$ (Wallace and Hobbs, 2006, Figure 3.9).
17. Pseudo-adiabatic lapse rate

Accounting for the latent heat of condensation during parcel motion (21) yields $\Gamma_p$, the

- saturated adiabatic lapse rate (most common)  \hspace{1cm} (22)
- pseudo-adiabatic lapse rate (PALR, most accurate)

\[
\Gamma_p = \frac{\Gamma_d}{1 + \frac{l_v d r_s}{c_p d T}} \tag{23}
\]

In the troposphere, $\Gamma_p \approx 5.0 \text{ K km}^{-1}$, close to the observed rate $\Gamma \approx 5.0 \text{ K km}^{-1}$!

Parcels ascend/descend adiabatically at MALR $\Gamma_m$ until saturation then latent heating causes $T$ to change at PALR $\Gamma_p$.  \hspace{1cm} (24)
18. Atmospheric stability

1) stable
\[ \Gamma_{\text{atmosphere (measured)}} < \Gamma_{\text{dry}} \]
unsaturated parcel at point O always returns to O

2) unstable
\[ \Gamma_{\text{atmosphere (measured)}} > \Gamma_{\text{dry}} \]
unsaturated parcel at point O rises further if lifted, sinks further if lowered.

Wallace and Hobbs Fig. 3.12
19. Atmospheric stability – conditional instability

\[ \Gamma_{\text{sat}} < \Gamma_{\text{atmosphere (measured)}} < \Gamma_{\text{dry}} \]

1. lift parcel from O to A along dry adiabat (stable)
2. above lifting condensation level (LCL) parcel follows saturated adiabat
3. above level of free convection (B) parcel is unstable

LCL and LFC depend on moisture content, temperatures, etc.

an external force is needed to initiate convection

Wallace and Hobbs Fig. 3.16
Figure 5. Tephigram (skew $T - \ln p$ plot) for estimating parcel trajectories (Wallace and Hobbs, 2006).
Figure 6. Measured sounding
http://weather.uwyo.edu/upperair/sounding.html).
Figure 6b. Measured sounding [http://weather.uwyo.edu/upperair/sounding.html](http://weather.uwyo.edu/upperair/sounding.html).