Chapter 15

Mechanical Waves

A wave is any disturbance from an equilibrium condition, which travels or propagates with time from one region of space to another.

A harmonic wave is a periodic wave in which each particle in the medium undergoes simple harmonic motion.

15.1 Types of Waves: Transverse and Longitudinal

A. Transverse Waves: The direction of motion of the particles in the medium is perpendicular to the direction of propagation of the wave. Examples include:
(1) waves on a rope
(2) electromagnetic waves

If one shakes a string up and down just once, the result is a single transverse wave pulse that travels along the length of the string.
If we shake the free end of a string in a repetitive, or periodic motion, then each particle in the string also undergoes periodic motion as the periodic transverse wave propagates along the string.

**B. Longitudinal Waves:** The direction of motion of the particles in the medium is parallel to the direction of propagation of the wave. Examples include:

1. sound

Consider a long tube filled with a fluid and a piston at the left end. If we push the piston just once to the right and then to the left, then a longitudinal wave pulse travels to the right along the fluid as shown in the figure below.
If we now push the piston to the right and to the left in simple harmonic motion along a line parallel to the length of the tube, then a periodic longitudinal wave propagates along the fluid parallel to the tube.

This motion forms regions in the fluid where the pressure and density are higher or lower than the equilibrium values. The regions in the fluid of increased pressure and density are called *compressions*, and the regions of reduced density and pressure are called *rarefactions*. 
Some waves have both transverse and longitudinal components as in the case of waves on the surface of a liquid.

\[ v = \frac{\text{distance}}{\text{time}} = \frac{\lambda}{T} = \lambda f \]
15.3 **Mathematical Description of a Wave**

A *sinusoidal wave* has a waveform equal to that of a sine (or cosine) wave.

15.10 (a) Another view of the wave at \( t = 0 \) in Fig. 15.9a. The vectors show the transverse velocity \( v_y \) and transverse acceleration \( a_y \) at several points on the string. (b) From \( t = 0 \) to \( t = 0.05T \), a particle at point 1 is displaced to point 1’, a particle at point 2 is displaced to point 2’, and so on.

A. Consider a **transverse sinusoidal waveform** in a rope of the kind shown in the figure above. The general
functional form of the displacement \( y(x,t) \) of a piece of rope at position \( x \) and time \( t \) is

\[
y(x,t) = A \sin(kx + \omega t)
\]

where \( A \) is the amplitude, \( k \) is the wave number, and \( \omega \) is the angular frequency as described in chapter 14. Also, remember from chapter 14 that

\[
k = \frac{2\pi}{\lambda} \quad \omega = 2\pi f = \frac{2\pi}{T}
\]

so that one may write the displacement \( y(x,t) \) as

\[
y(x,t) = A \sin \left( 2\pi \left( \frac{x + t}{\lambda} \right) \right)
\]

To find the speed \( v \) of wave propagation, one can simply say that the wave propagates a distance equal to one wavelength \( \lambda \) in a time interval equal one period \( T \). Hence,

\[
v = \frac{distance}{time}
\]

\[
v = \frac{\lambda}{T} = \lambda f
\]
so that one may also write

\[ y(x, t) = A \sin \left( \frac{2\pi}{\lambda} \left( x + \frac{\lambda t}{T} \right) \right) \]

Note that the disturbance \( y(x, t) \) is a function of \( x \pm vt \).

**B. Transverse Velocity and Acceleration of particle on the rope**

Consider the waveform on a rope described by

\[ y(x, t) = A \sin(kx - \omega t) \]

What are the transverse velocity \( v_y \) and acceleration \( a_y \) of a piece of rope at point \( x \) down the rope and time \( t \)? Well,

\[ v_y = \frac{dy}{dt} \quad \text{(holding } x \text{ constant)} \]

\[ v_y = (-\omega) A \cos(kx - \omega t) \]
note carefully that the maximum value of the transverse speed of a piece of rope equals (setting the cosine term equal to one)

\[ v_{y, \text{max}} = \omega A \]

which is the same result obtained in chapter 14 where we said after using conservation of mechanical energy that

\[ \frac{1}{2} m v_{\text{max}}^2 = \frac{1}{2} k A^2 \quad \Rightarrow \quad v_{\text{max}} = \sqrt{\frac{k}{m}} A = \omega A \quad \text{same result!} \]

We know that a piece of rope will attain its maximum speed when it is moving through the equilibrium position. That is, a piece of rope that has a transverse wave propagating through it undergoes simple harmonic motion!

Be very careful here!!! Do not confuse the speed of propagation of the wave \( v = \lambda f \) with the transverse speed \( v_y \) of a piece of rope!!

As for the acceleration of a piece of rope at position \( x \) and time \( t \), then

\[ a_y = \frac{d(v_y)}{dt} \quad \text{(holding} \ x \ \text{constant)} \]

\[ a_y = (-\omega)(-\omega)A \{ -\sin(kx - \omega t) \} \]
\[ a_y = -\omega^2 A \sin(kx - \omega t) \]

\[ a_y = -\omega^2 y \]

This result is identical to the result obtained in chapter 14. Note in addition that the maximum value of the magnitude of the acceleration of a piece of rope occurs at the amplitudes, i.e.,

\[ a_{y,\text{max}} = \omega^2 A \]

By the way, the wave function \( y(x, t) = A \sin(kx - \omega t) \) satisfies an equation called a “wave equation”, given by

\[ \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \]

Any function of the form \( f(x \mp vt) \) satisfies the wave equation. We will learn in chapter 32 that when an electromagnetic wave propagates through vacuum, both the electric field \( \vec{E} \) and the magnetic field \( \vec{B} \) obey a wave equation quite similar to the last equation above.
15.4 Speed of a Transverse Wave on a String

The speed \( v \) with which a wave propagates in a rope of mass per unit length \( \mu \) and under tension \( F \) is given by

\[
v = \sqrt{\frac{\text{Tension}}{\mu}}
\]

**Derivation:**
Consider a small segment \( \Delta x \) and mass \( \Delta m \) of a very long string under tension \( F \). The tension (force) at each end of the string is tangent to the string at the point of application.
The string to the right of the small segment exerts a force $F_2$ on the segment. The string to the left of the small segment exerts a force $F_1$ on the segment. There is a net vertical force on the segment of string, but not a net horizontal force since the motion of the string is vertical only.

Apply Newton’s Second Law of motion:

$$\sum F_y = (\Delta m) \frac{\partial^2 y}{\partial t^2}$$

but

$$\sum F_y = F_{2y} - F_{1y} = F \tan \theta_2 - F \tan \theta_1$$

where $\theta_2$ is the angle that $F_2$ makes with the horizontal, and $\theta_1$ is the angle that $F_1$ makes with the horizontal. But the $\tan(\theta) = \text{slope of the tangent line to the string (curve)}$ and this is also equal to $\frac{\partial y}{\partial x}$. Hence

$$\sum F_y = F \tan \theta_2 - F \tan \theta_1 = F \left( \frac{\partial y}{\partial x} \right)_{x+\Delta x} - F \left( \frac{\partial y}{\partial x} \right)_{x}$$

note that in the limit as $\Delta x \to 0$, 

\[
\left( \frac{\partial y}{\partial x} \right)_{x+\Delta x} - \left( \frac{\partial y}{\partial x} \right)_{x} = \frac{\partial^2 y}{\partial x^2}(\Delta x)
\]

so that

\[
\sum F_y = F\left( \frac{\partial^2 y}{\partial x^2} \right)(\Delta x)
\]

Therefore,

\[
\sum F_y = (\Delta m) \frac{\partial^2 y}{\partial t^2} = F\left( \frac{\partial^2 y}{\partial x^2} \right)(\Delta x)
\]

use \( \Delta m = \mu \Delta x \), where \( \mu \) is defined as the mass per unit length of the string, to obtain

\[
(\mu \Delta x) \frac{\partial^2 y}{\partial t^2} = F\left( \frac{\partial^2 y}{\partial x^2} \right)(\Delta x)
\]

\[
\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F} \frac{\partial^2 y}{\partial t^2}
\]

Finally, comparing this last expression with the general expression for a wave equation, \( \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \) yields,
15.6 Wave Interference, and the Superposition Principle

If two or more waves are traveling in a medium and interfere, the resultant wave is found by adding the displacements of the individual waves.

Interference of Waves

Use transparencies to discuss concept of phase difference.

A. Constructive Interference of 2 waves of the same frequency:
Constructive interference will result if the phase difference $\phi$ between the 2 interfering waves is $\phi = 0, 2\pi, 4\pi, 6\pi,...$

B. Destructive Interference of 2 waves of the same frequency:

Destructive interference will result if the phase difference $\phi$ between the 2 interfering waves is $\phi = \pi, 3\pi, 5\pi,...$
15.8 Normal Modes of a String

The string has various natural patterns of vibration, called *normal modes*, and each normal mode has a characteristic frequency.

What are the allowed frequencies of vibration?

15.26 The first four normal modes of a string fixed at both ends. (Compare these to the photographs in Fig. 15.23.)

(a) $n = 1$: fundamental frequency, $f_1$

$$L = \frac{\lambda}{2}, \quad n = 1$$

(b) $n = 2$: second harmonic, $f_2$ (first overtone)

$$L = 2\left(\frac{\lambda}{2}\right), \quad n = 2$$

(c) $n = 3$: third harmonic, $f_3$ (second overtone)

$$L = 3\left(\frac{\lambda}{2}\right), \quad n = 3$$

(d) $n = 4$: fourth harmonic, $f_4$ (third overtone)

$$L = 4\left(\frac{\lambda}{2}\right), \quad n = 4$$
In general, \( L = n \left( \frac{\lambda}{2} \right) \) where \( n = 1, 2, 3, 4, \ldots \), and so

\[
\lambda = \frac{2L}{n}
\]

and the allowed frequencies of vibration \( f = \frac{v}{\lambda} \) are equal to

\[
f_n = n \left( \frac{v}{2L} \right) \quad (n = 1, 2, 3, 4,\ldots)
\]

Because \( v = \sqrt{\frac{Tension}{\mu}} \), one can express the natural frequencies of vibration of a stretched string as

\[
f_n = \frac{n}{2L} \sqrt{\frac{Tension}{\mu}} \quad (n = 1, 2, 3, 4,\ldots)
\]

The lowest allowed natural frequency of vibration is called the fundamental frequency. Any integer multiple of the fundamental frequency is called a harmonic. Thus,
\[ f_1 \equiv \text{fundamental frequency} \quad \Rightarrow \quad f_1 = \frac{1}{2L} \sqrt{\frac{\text{tension}}{\mu}} \]

\[ f_2 = 2f_1 \equiv \text{second harmonic (or first overtone)} \]

\[ f_3 = 3f_1 \equiv \text{third harmonic (or second overtone)} \]

and so on. All the even and odd harmonics are present.

### 15.7 Mathematical Representation of Standing Waves

Consider two waves of the same frequency \( \omega \) and same amplitude \( A \) traveling in opposite directions.

**wave 1:**
\[ y_1 = A \sin(kx - \omega t) \]

**wave 2:**
\[ y_2 = A \sin(kx + \omega t) \]

Note that wave 1 travels to the right, while wave 2 travels to the left.

What will the resultant wave be? The resultant wave will be a “standing wave” or “blinking wave” as I like to call them. Applying the superposition principle yields

\[ y = y_1 + y_2 \]

\[ y = A \sin(kx - \omega t) + A \sin(kx + \omega t) \]
\[ y = A [ \sin(kx - \omega t) + \sin(kx + \omega t) ] \]

one learns in trigonometry that

\[
\sin(kx - \omega t) = \sin(kx)\cos(\omega t) - \sin(\omega t)\cos(kx)
\]

\[
\sin(kx + \omega t) = \sin(kx)\cos(\omega t) + \sin(\omega t)\cos(kx)
\]

so that the resultant wave is

\[ y = 2A \sin(kx)\cos(\omega t) \quad \text{a “standing wave”} \]

Note carefully that this is NOT a traveling wave because \( y \) is not a function of \( x \pm vt \).

The sine term in the standing wave describes the \textit{spatial dependence}, while the cosine term describes the \textit{time dependence} of the standing wave.

\textbf{Questions:}

1. At positions or values of \( x \) will the disturbance be \textit{zero} at \textit{all times}? These positions are called \textit{nodes}.

\[
\sin(kx) = 0
\]

\[
kx = 0, \pi, 2\pi, 3\pi, \ldots, n\pi \quad \text{where} \quad (n = 0, 1, 2, 3, \ldots)
\]
\[ \frac{2\pi}{\lambda} x = n\pi \]

\[ x = n \frac{\lambda}{2} \]

(nodes)

that is, when \( x = 0, \frac{1}{2}\lambda, \lambda, \frac{3}{2}\lambda, 2\lambda, \ldots \)

2. At what points \( x \) will the disturbance be a maximum at all times? These positions are called \textit{antinodes}.

\[ \sin(kx) = \pm 1 \]

\[ kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{n\pi}{2} \]

where \( n = 1, 3, 5, 7, \ldots \)

\[ \frac{2\pi}{\lambda} x = \frac{n\pi}{2} \]

\[ x = n \frac{\lambda}{4} \]

(antinodes)
3. At what times \( t \) will the disturbance be \textit{zero} for \textit{all} values of \( x \)?

\[
\cos(\omega t) = 0
\]

\[
\omega t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{n\pi}{2}
\]

where \( n = 1, 3, 5, 7, \ldots \)

\[
\frac{2\pi}{T} t = \frac{n\pi}{2}
\]

where \( T = \text{period} \)

\[
t = n \frac{T}{4}
\]

that is, zero disturbance for all positions of \( x \) at times

\[
t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}, \ldots
\]

The figure below shows the shape of a string at 2 different instants:

![String Diagram](image)
15.5 **Energy in Wave Motion**
Waves transport energy from one place to another. To produce a wave, we have to exert a force on the wave medium, thus *doing work on the system*. As the wave propagates, each portion of the medium does work on the adjoining portion. In this way, a wave transports energy from one region of space to another.

**A. Waves Propagating Through an Elastic Medium**

Consider a transverse wave traveling from left to right on a string. Now consider a specific point on the string. The string to the left of this point exerts a force (in the $y$-direction) to the right of it given by $F_y = -F \tan \theta$, where $\tan \theta = \frac{\Delta y}{\Delta x}$, so that $F_y = -F \frac{\partial y}{\partial x}$. This force does work on the portion of string to the right of it and therefore transfers energy to it. The corresponding *power* (rate of doing work) at this point in the string is $P = F_y v_y$, that is, the transverse force multiplied by the transverse velocity.
\[ P(x,t) = F_y(x,t) \] \[ v_y(x,t) = \left( -F \frac{\partial y}{\partial x} \right) \left( \frac{\partial y}{\partial t} \right) \]

using \( y(x,t) = A \sin(kx - \omega t) \), then

\[
v_y = \frac{\partial y}{\partial t} = (-\omega) A \cos(kx - \omega t)
\]

\[
\frac{\partial y}{\partial x} = (k) A \cos(kx - \omega t)
\]

so that

\[
P(x,t) = \left[ -FkA \cos(kx - \omega t) \right] \left[ -\omega A \cos(kx - \omega t) \right]
\]

\[ P(x,t) = F k \omega A^2 \cos^2(kx - \omega t) \]

using \( \omega = k \nu \) and \( \nu = \sqrt{\frac{F}{\mu}} \) yields

\[
P(x,t) = \mu F \omega^2 A^2 \cos^2(kx - \omega t)
\]

**Comments:**

1. Note that the **instantaneous power** is never negative. It is either positive (corresponding to energy flow in the positive x-direction) or it is zero (at points where there is no energy transfer).
2. The energy transported by a wave is proportional to the square of the frequency and to the square of its amplitude. This is true of mechanical waves.

3. The maximum value of the instantaneous rate of energy transfer is

\[ P_{\text{maximum}} = \sqrt{\mu F \omega^2 A^2} \]

4. The average power \( P_{\text{average}} \) transported by the wave is the rate of energy transferred. The average of the cosine squared term over a whole cycle is always \( \frac{1}{2} \) so that

\[ P_{\text{average}} = \frac{1}{2} \sqrt{\mu F \omega^2 A^2} \]

15.16 The instantaneous power \( P(x, t) \) in a sinusoidal wave as given by Eq. (15.23), shown as a function of time at coordinate \( x = 0 \). The power is never negative, which means that energy never flows opposite to the direction of wave propagation.
5. The intensity transported $I$ is the average power transported per unit cross sectional area so that

$$I = \frac{P_{\text{average}}}{\text{Area}} = \frac{\sqrt{\mu F \omega^2 A^2}}{2(\text{Area})}$$

If waves spread out equally in all directions from a source, then the intensity $I$ of the wave at a distance $r$ from the source is inversely proportional to $r^2$, because the cross sectional area is $4\pi r^2$. Thus

$$I = \frac{P_{\text{average}}}{\text{Area}} = \frac{P}{4\pi r^2}$$

so from conservation of energy, the power output $P$ from the source is equal at the surface with radius $r_1$ and at the surface with radius $r_2$ if no energy is absorbed between the two surfaces. Thus

$$\text{Power} = 4\pi r_1^2 I_1 = 4\pi r_2^2 I_2$$

15.17 The greater the distance from a wave source, the greater the area over which the wave power is distributed and the smaller the wave intensity.
B. Electromagnetic Waves Propagating through Vacuum

We will learn in chapter 32 that the intensity transported by an electromagnetic wave in vacuum is proportional to the square of the electromagnetic field amplitude, but is independent of the frequency of the wave.