1. Two very fast trains are traveling (each at 0.9 c) on parallel tracks in opposite directions. There is an observer on each train. The first observer is traveling in the +x direction, and the second observer is traveling in the -x direction. The observers synchronize their clocks to $t = 0$ as they pass each other at $x = 0$.

As the observers pass, the first observer shoots a bullet at a speed of 0.5 c (relative to the first observer) at a target 20 m away toward the back of his train.

When does the bullet reach the target according to (a) the first observer and (b) the second observer?

What is the position of the target when the bullet reaches it, according to (a) the first observer and (b) the second observer?

The first observer is at rest with respect to the target so everything’s simple. The target is at $x = -20$ m, and the bullet reaches it at $t = 20/(0.5c) = 13.3 \times 10^{-8}$ s.

For the second observer, the target is moving at a velocity $v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{1.8}{1.81} c$.

For the second observer, $t', x'$ are set by

$$ct' = \gamma(ct + \beta x)$$
$$x' = \gamma(x + \beta ct)$$

where $\beta = \frac{1.8}{1.81}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

2. A pion (rest mass 139 MeV/c$^2$) is traveling at 0.7 c along the +x axis when it decays to a muon (rest mass 100 MeV/c$^2$) and a photon. The photon 3-momentum is along the +y axis. Find the energy of the muon.

The four momentum of the pion is

$$p_{\text{pion}}^\mu = (m_{\text{pion}} \gamma_{\text{pion}}, m_{\text{pion}} \gamma_{\text{pion}} v_{\text{pion}}, 0, 0)$$

where $v_{\text{pion}} = 0.7c$, $\gamma_{\text{pion}} = \frac{1}{\sqrt{1-0.7^2}} = 1.4$

The four momentum of the photon is

$$p_{\text{photon}}^\mu = (E_{\text{photon}}, 0, E_{\text{photon}}, 0)$$

By the conservation of 4-momentum, the muon 4-momentum is

$$p_{\text{muon}}^\mu = (m_{\text{pion}} \gamma_{\text{pion}} - E_{\text{photon}}, -m_{\text{pion}} \gamma_{\text{pion}} v_{\text{pion}}, 0, -E_{\text{photon}}, 0)$$

We require this to satisfy $E_{\text{muon}}^2 - p_{\text{muon}}^2 = m_{\text{muon}}^2$. So

$$(m_{\text{pion}} \gamma_{\text{pion}} - E_{\text{photon}})^2 - (m_{\text{pion}} \gamma_{\text{pion}} v_{\text{pion}})^2 - (E_{\text{photon}})^2 = m_{\text{muon}}^2$$

which can be simplified to

$$(m_{\text{pion}})^2 - 2m_{\text{pion}} \gamma_{\text{pion}} E_{\text{photon}} = m_{\text{muon}}^2$$
which allows us to solve for $E$:

$$E_{\text{photon}} = \frac{(m_{\text{pion}})^2 - m_{\text{muon}}^2}{2m_{\text{pion}}\gamma_{\text{pion}}} = \frac{139^2 - 100^2}{2(139)(1.4)} \text{MeV} = 23.9 \text{MeV} \quad (8)$$

the muon energy is thus

$$E_{\text{muon}} = 139(1.4) \text{MeV} - 23.9 \text{MeV} = 170.7 \text{MeV} \quad (9)$$