What does probability mean??

What does it mean to say:

- The *probability* of rain tomorrow is .2.
- The *probability* that a coin toss will land heads up is $\frac{1}{2}$.
- The *probability* that humans will survive to the year 3000 is .8.

Is the word “probability” interpreted the same way in all of these?
Two basic interpretations of probability (Summary box, p. 226)

Interpretation 1: Relative frequency
- Used for repeatable circumstances
- The probability of an outcome is the proportion of time that outcome does or will happen in the long run.

Interpretation 2: Personal probability (subjective)
- Most useful for one-time events
- The probability of an outcome is the degree to which an individual believes it will happen.
Two methods for determining relative frequency probability

1. **Make an assumption** about the physical world *or*
2. **Observe** the *relative frequency* of an outcome *over* many repetitions. “Repetitions” can be:
   a. *Over time*, such as how often a flight is late
   b. *Over individuals*, by measuring a representative sample from a larger population and observing the *relative frequency* of an outcome or category of interest, such as the probability that a randomly selected person is left-handed.
Personal Probability: Especially useful for one-time only events

- The *personal probability* of an outcome is *the degree to which* an individual *believes* it will happen.
- Could be different for two different people.
Section 7.3: Probability definitions and relationships

We will use 2 examples to illustrate:

1. Daily 3 lottery winning number.
   Outcome = 3 digit number, from 000 to 999

2. Choice of 3 parking lots on campus. You try Lot 1, if full then Lot 2, if full then Lot 3.
   Lot 1 works 30% of the time, you aren’t late
   Lot 2 works 50% of the time, you are late
   Lot 3 always works, so you park there 20% of the time, and when you do, you are very late!
Definitions of *Sample space* and *Simple event*

The **sample space** $S$ for a random circumstance is the collection of unique, non-overlapping outcomes.

A **simple event** is *one outcome* in the sample space.

Ex 1: $S = \{000, 001, 002, \ldots, 999\}$
   
   One simple event: 659
   
   There are 1000 simple events.

Ex 2: $S = \{\text{Lot 1, Lot 2, Lot 3}\}$
   
   One simple event: Lot 2
   
   There are 3 simple events.
Definition And Notation For Events

- **Event:** *any subset* of the sample space.
- **Compound event:** More than one simple event.

Notation for events: A, B, C, etc.

Ex 1: A = *winning number begins with 00*
   - A = {000, 001, 002, 003, ..., 009}
   - B = *all same digits* = {000, 111, ..., 999}

Ex 2: A = *late for class* = {Lot 2, Lot 3}
Notation: $P(A) =$ probability of the event $A$

Rules: Probabilities are always assigned to *simple events* such that these 2 rules must hold:

1. $0 \leq P(A) \leq 1$ for each simple event $A$
2. The *sum* of probabilities of *all* simple events in the sample space is 1.

The *probability of any event* is the sum of probabilities for the simple events that are part of it.
Special Case: Assigning Probabilities to Equally Likely Simple Events

Equally Likely Simple Events
If there are $k$ simple events in the sample space and they are all equally likely, then the probability of the occurrence of each one is $1/k$.

Example:
Roll a fair die with numbers 1 to 6, so $k = 6$. $S = \{1, 2, 3, 4, 5, 6\}$, each with probability $1/6$. 
Example: California Daily 3 Lottery

Random Circumstance:
A three-digit winning lottery number is selected.

Sample Space:
\{000, 001, 002, 003, \ldots, 997, 998, 999\}.
There are 1000 simple events.

Physical assumption: all three-digit numbers are equally likely.

Probabilities for Simple Event: Probability that any specific three-digit number is a winner is 1/1000.
Examples of Complex Events:

- **Event A** = last digit is a 9 = \{009, 019, \ldots, 999\}.
  
- \(P(A) = \frac{100}{1000} = \frac{1}{10}\) (there are 100 simple events)

- **Event B** = three digits are all the same
  \{000, 111, 222, 333, 444, 555, 666, 777, 888, 999\}.

- \(P(B) = \frac{10}{1000} = \frac{1}{100}\) (there are 10 simple events)
**Example 2:**
Simple events are *not* equally likely

<table>
<thead>
<tr>
<th>Simple Event</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Park in Lot 1</td>
<td>.30</td>
</tr>
<tr>
<td>Park in Lot 2</td>
<td>.50</td>
</tr>
<tr>
<td>Park in Lot 3</td>
<td>.20</td>
</tr>
</tbody>
</table>

Note that these sum to 1

Event $A = \text{late for class} = \{\text{Lot 2, Lot 3}\}$

$P(A) = .50 + .20 = .70$
Probability in daily language:

People express probabilities as percents, proportions, probabilities. These are all equivalent:

- United flight 436 from SNA to Chicago arrives on time \textit{81 percent} of the time.
- The \textit{proportion} of time United flight 436 arrives on time is \textit{.81}.
- The \textit{probability} that United flight 436 will arrive on time is \textit{.81}.
RELATIONSHIPS BETWEEN EVENTS

• Defined for events in the *same random circumstance only*:
  – Complement of an event
    • The event doesn’t happen
  – Mutually exclusive events = disjoint events
    • Two events don’t overlap

• Defined for events in the same *or different random circumstances*:
  – Independent events
  – Conditional events
Definition and Rule 1 (apply to events in the same random circumstance):

**Definition**: One event is the *complement* of another event if:
- They have no simple events in common, AND
- They cover all simple events

**Notation**: The complement of \( A \) is \( A^C \)

**RULE 1**: \( P(A^C) = 1 - P(A) \)

Ex 2: *Random circumstance* = parking on one day

\( A \) = late for class, \( A^C \) = on time

\( P(A) = .70 \), so \( P(A^C) = 1 - .70 = .30 \)
Complementary Events, Continued

Rule 1: \( P(A) + P(A^c) = 1 \)

Example: *Daily 3 Lottery*
- \( A \) = player buying single ticket wins
- \( A^c \) = player does not win
- \( P(A) = 1/1000 \) so \( P(A^c) = 999/1000 \)

Example: *On-time flights*
- \( A \) = flight you are taking will be on time
- \( A^c \) = flight will be late
- Suppose \( P(A) = .81 \), then \( P(A^c) = 1 - .81 = .19 \).
Mutually Exclusive Events

Two events are **mutually exclusive**, or equivalently **disjoint**, if they do not contain *any* of the same simple events (outcomes). (Applies in *same random circumstance.*)

**Example: Daily 3 Lottery**

A = all three digits are the same (000, 111, etc.)
B = the number starts with 13 (130, 131, etc.)

The events A and B are **mutually exclusive** (disjoint), but they are **not complementary**.
(No overlap, but *don’t cover all possibilities.*)
Independent and Dependent Events

• Two events are independent of each other if knowing that one will occur (or has occurred) does not change the probability that the other occurs.
• Two events are dependent if knowing that one will occur (or has occurred) changes the probability that the other occurs.

The definitions can apply either ... to events within the same random circumstance or to events from two separate random circumstances.
EXAMPLE OF INDEPENDENT EVENTS

- Events in the *same random circumstance*:
  - Daily 3 lottery on the *same* draw
    - \( A = \text{first digit is 0} \)
    - \( B = \text{last digit is 9} \quad P(B) = 1/10 \)
  - Knowing first digit is 0, \( P(B) \) is still 1/10.

- Events in *different random circumstances*:
  - Daily 3 lottery on *different* draws
    - \( A = \text{today’s winning number is 191} \)
    - \( B = \text{tomorrow’s winning number is 875} \)
  - Knowing today’s # was 191, \( P(B) \) is still 1/1000
Mutually exclusive or independent?

- If two events are mutually exclusive (disjoint), they cannot be independent:
  - If disjoint, then knowing A occurs means \( P(B) = 0 \)
  - In independent, knowing A occurs gives no knowledge of \( P(B) \)

Example of mutually exclusive (disjoint):
  \( A = \text{today’s winning number is 191}, \)  
  \( B = \text{today’s winning number is 875} \)

Example of independent:
  \( A = \text{today’s winning number is 191} \)  
  \( B = \text{tomorrow’s winning number is 875} \)
Conditional Probabilities

The **conditional probability** of the event $B$, **given** that the event $A$ has occurred or will occur, is the long-run relative frequency with which event $B$ occurs when circumstances are such that $A$ also occurs; written as $P(B|A)$.

$P(B) = \text{unconditional probability event } B \text{ occurs.}$

$P(B|A) = \text{“probability of } B \text{ given } A\text{”} = \text{conditional probability event } B \text{ occurs given that we know } A \text{ has occurred or will occur.}$
EXAMPLE OF CONDITIONAL PROBABILITY

**TABLE 2.3**  ■ Nighttime Lighting in Infancy and Eyesight

<table>
<thead>
<tr>
<th>Slept with:</th>
<th>No Myopia</th>
<th>Myopia</th>
<th>High Myopia</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Darkness</td>
<td>155 (90%)</td>
<td>15 (9%)</td>
<td>2 (1%)</td>
<td>172</td>
</tr>
<tr>
<td>Nightlight</td>
<td>153 (66%)</td>
<td>72 (31%)</td>
<td>7 (3%)</td>
<td>232</td>
</tr>
<tr>
<td>Full Light</td>
<td>34 (45%)</td>
<td>36 (48%)</td>
<td>5 (7%)</td>
<td>75</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>342 (71%)</td>
<td>123 (26%)</td>
<td>14 (3%)</td>
<td>479</td>
</tr>
</tbody>
</table>

Random circumstance: Observe one randomly selected child

A = child slept in darkness as infant  [Use “total” column.]

\[
P(A) = \frac{172}{479} = .36
\]

B = child did not develop myopia  [Use “total” row]

\[
P(B) = \frac{342}{479} = .71
\]

\[
P(B|A) = P(\text{no myopia} | \text{slept in dark}) \quad \text{[Use “darkness” row]}
\]

\[
= \frac{155}{172} = .90 \neq P(B)
\]
NOTES ABOUT CONDITIONAL PROBABILITY

1. $P(B|A)$ generally does not equal $P(B)$.  
2. $P(B|A) = P(B)$ only when $A$ and $B$ are independent events.  
3. In Chapter 4, we were actually testing if two types of events were independent.  
4. Conditional probabilities are similar to row and column proportions (percents) in contingency tables. Myopia example on previous page: $P(\text{no myopia} \mid \text{dark})$ is the row proportion for no myopia.