Problem Solving
Types of Problems Solving

- **Well-defined** problems involving
  - An initial or starting state
  - A clear end or goal state
  - A set of allowable moves or actions to change states

- Combinatorial or crypto-arithmetic problems
  - Widely studied in artificial intelligence and cognitive psychology
    - Towers of Hanoi, Cannibals and Missionaries, ...
    - Traveling Salesperson Problems and other very large state space optimization problems

- Optimization problems involving inherent uncertainty
  - Optimal stopping, bandit problems, ...

- **Ill-defined** problems
  - Not clear what the problem states or actions are
    - e.g., insight problems
Cannibals and Missionaries

  - Help the 3 cannibals and 3 missionaries to move to the other side of the lake. Note that when there are more cannibals on one side of the lake than missionaries, the cannibals eat them. So you need to think logically and strategically in order to get them all across to the other side safely.
Theseus and the Minotaur

- Move Theseus (red circle) to the exit without being eaten by the Minotaur (black circle)
- For each move that Theseus makes, the Minotaur makes two moves.
  - He always tries to get closer to Theseus
  - If he can move one square horizontally and get closer, he will
  - If he can’t move horizontally, then he will try to move vertically.
- The key to solving these mazes is that the Minotaur follows rules
  - He doesn’t look ahead more than one turn
  - He will choose a horizontal move before a vertical move.
Towers of Hanoi

- Have to move the disc from the left peg to the right peg, with the rules that
  - A wider disc cannot be placed on top of a narrower disc
  - As few moves as possible must be used
These problems have problem spaces (aka state spaces) with initial and end goals, and connections between them that follow the rules.
Newell and Simon (1972) developed a problem space theory as a general account of how people solve these sorts of problems.

- Views problem solving as serial search of the states, moving from the initial state to the goal using heuristics to generate actions.

Various heuristics have been proposed, including:

- Trial and error
- Hill climbing
- Means-end analysis
Trial and Error

• Thorndike (1932) studied how cats solved problems in puzzle boxes
  – In order to escape the puzzle box shown, the cat has to perform three different actions
    – press a pedal, pull on a string, and push a bar up or down
  • Found that many animals search by trial and error
    – Initially behaved impulsively and apparently randomly, but eventually found solutions
    – After many trials in puzzle box, solution time decreases.
Hill Climbing

- Find some measure of the distance between your present state and the end state.
  - Take a step in the direction that most reduces that distance
Hill Climbing

- Using the hill-climbing heuristic, at each step, you choose a path that moves you closer to the goal
  - This strategy can fail when steps are required that initially will move away from the goal
  - Can lead to a suboptimal solution, known as a local maximum
    - Might explain clinical cases of perseverance from frontal lobe damage

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Means-end Analysis

- Means-end analysis works by applying a set of steps
  - Set up a goal
  - Look for a difference between current state and goal or subgoal state
  - Find an operator to reduce this difference.
    - One operator is the setting of a new subgoal
  - Apply the operator
  - Repeat until final goal is achieved

(a) Subgoal 1: Free up large disc.

(b) Subgoal 2: Free up third peg.

(c) Subgoal 3: Move large disc onto third peg.
Combinatorially Larger Problems

- Human performance on problems like the Towers of Hanoi has been extensively studied, but it doesn’t seem like the sort of ‘real-world’ problem people solve all the time
  - People certainly do not seem to solve the problem in an automatic or effortless way
- More recently, how people solve optimization problems with extremely large state spaces have been studied
  - Many of these are visually presented
    - Traveling Salesperson Problem, Minimum Spanning Tree Problem, ...
Minimum Spanning Tree Problem

- Given a set of points, connect them with straight edges so that
  - You can get from every point to every other point (spanning)
  - The total length of the edges is minimal
Traveling Salesperson Problem

- Given a set of points, connect them with straight edges so that
  - You start and finish at the same point, connecting all of the dots (form a tour)
  - The total length of the tour edges is minimal
Variants of Perceptual Optimization Problems

- This is an emerging area, and many variants are being studied
  - Different metrics, using obstacles, 3D environments...
Properties of and Performance on TSPs

- TSP problems are different from Towers of Hanoi type problems in the sense that
  - their solutions are not computationally tractable (i.e., no solution method exists that is guaranteed to find the best answer in a reasonable length of time)
  - they seem more closely related to common real-world problems, like deciding how to get from A to B
- Empirical studies show that
  - people seem to be able to solve them relatively well and relatively easily
  - individual differences in solution quality are related to measures of intelligence
- There is some suggestion there is a relationship between these types of problems and some standard neuropsychological tests
The Trail Making Test (TMT)

- The TMT is an important neuropsychological test
- Connect the dots in order
  - 1, 2, 3, 4, 5, ... or 1, A, 2, B, 3, C .... depending on form
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Der Zahlen-Verbindungs Test (ZVT)

- A trail-making test, requiring subjects to connect the circles in order
- Takes about a minute to complete, and correlates moderately (around r=0.4) with more time-consuming measures of psychometric intelligence
Optimal Stopping Problem

- The ‘recreational mathematics’ problem known as the Secretary Problem is also sometimes (even less appropriately) known as the “dowry problem” or the “beauty contest problem”
- It is the prototypical optimal stopping problem, and involves making a choice in a decision sequence under uncertainty
- The rules of the problem are:
  - you have to select the best from a known number of sequentially presented alternatives
  - for each alternative, you are only told its rank order in relation to those already seen
  - if you wish to select an alternative, you must do it at the time it is presented
Two Versions of the Secretary Problem

- In the full information version, you see the actual numbers (one at a time)
  - e.g., 66, 4, 85, 93, 68, 76, 74, 39, 67, 17
  - The optimal solution is to pick the first number that is the maximum seen so far and above a threshold that is specific to each position in the sequence
  - i.e., 92, 91, 89, 88, 86, 82, 78, 69, 50, 0

- In the rank information version, you see just the rank of the current number relative to previous ones (one at a time)
  - e.g., 1, 2, 1, 1, 3, 3, 4, 7, 7, 9
  - The optimal solution is to wait for the first 37% of the sequence, then pick the next 1st ranked option (if there is one)
  - i.e., Wait, Wait, Wait, Yes
Real World Optimal Stopping

- Optimal stopping problems like secretary problems are (more or less) analogous to many real-world sequential choice tasks
  - Any problem where you can hold out longer hoping for a better outcome, but cannot go back to earlier options
    - e.g., Stopping to buy gas on long drives through sparsely populated areas
    - e.g., Choosing a partner through a sequence of dating
Bandit Problems

• In bandit problems you have some number of choices to make, between the same small set of alternatives
  – Each alternative has some fixed, but unknown rate of giving a reward when you choose it
  – Your goal is to get the maximum number of rewards from your choices
Psychology of Bandit Problems

- Bandit problems are psychologically interesting because they capture the tension between exploration and exploitation in decision-making.
- The right thing to do is move from an early exploration strategy to a late exploitation strategy.
- But the balance between exploration and exploitation depends on the observed pattern of successes and failures and is sensitive to the nature of the environment.
- Real world examples of bandit problems include (more or less) any situation where you are choosing between staying with your current choice or switching to a new one:
  - e.g., loyalty to a familiar restaurants chain that is reasonably good, versus exploring a new one that might be better or worse.
  - e.g., the “Iowa Gambling Task”, which is used as a clinical measure of risk-seeking behavior is similar to a bandit problem.
Ill-Defined Problems

- Ill-defined problems are ones where solutions are generally clear and agreed once they are found, but it is not obvious that they have a start and end state, and a way of moving from one to the other.

- According to Gestalt psychologists, the problem may be solved suddenly by ‘seeing’ the problem differently.
  - Often requires developing a suitable representation, which is given as part of the statement in well-defined problems.

- A special case of an ill-defined problem is known as the **insight problem**, to which, despite all the unknowns, the answer seems to come all of a sudden in a flash of understanding.
Mutilated Checkerboard Problem

- A checkerboard has two corner tiles, diagonally opposite, removed, leaving 62 squares
  - Can 31 dominos (each 2 x 1 tiles in size) be arranged flush on this mutilated board?
  - Insight is “no”, because each domino covers 1 black and 1 white square, but both removed squares are the same color
Banana Problem

- Kohler (1945) observed that chimpanzees appeared to have an insight into the problem before solving it
  - No obvious process of trial and error was observed
  - A “Eureka!” moment
Empirical Evidence for Insight

- Metcalfe and Weibe (1987) studied the difference between well-defined and insight problems experimentally
  - Non-insight problem (algebra)
    - factor $16y^2 - 40yz + 25z^2$
  - Insight problem
    - A prisoner was attempting escape from a tower. He found in his cell a rope which was half long enough to permit him to reach the ground safely. He divided the rope in half and tied the two parts together and escaped. How could he have done this?
      - [Answer: They unraveled the rope lengthwise and tied the remaining strands together]
Metcalfe and Weibe Results

- Subjects “feelings of knowing” (beforehand) only predicted eventual success of solving the problem for non-insight problems.
- At 15 seconds intervals, subjects rated how close they felt to solving the problem:
  - 1=cold
    - (nowhere close to solution)
  - 7=hot
    - (problem is virtually solved)
- Found a gradual increase for the algebra problem, but a sudden increase for the insight problem
Rebus Problems

- A generic type of insight problem are known as “rebus” problems
- These are sub-word level rebus problems, taken from MacGregor and Cunningham (2009)
  - e.g., “big Bad wolf”, “capital punishment”, “little league”, ...

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### Rebus Problems

- These are sub-word level rebus problems
  - e.g., “go stand in the corner”, “split personality”, “too big to ignore”

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**Solutions:**
- The rebus for “go stand in the corner” is **A P P E S L U A**.
- The rebus for “split personality” is **person**.
- The rebus for “too big to ignore” is **big big ignore ignore**.
- The rebus for “suit and search” is **t s u i h t s**.
Rebus Problems

- These are sub-word level rebus problems
  - e.g., “somewhere over the rainbow”, “a home away from home”, “beating about the bush”,

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MacGregor and Cunningham Results

- MacGregor and Cunningham found that the proportion of people finding a solution
  - increased from sub-word to word- to supra-word problems
  - Increased if fewer “restructurings” were needed
Summary

- Well-defined problems with clear and relatively small state spaces have been widely studied
  - Useful theories for how people solve these, including various heuristic search approaches
- Well-defined problems with much larger state spaces, or incorporating uncertainty, are now being studied
  - Some of the same heuristics apply, but new theories are needed to account for how people deal with the uncertainty
- Ill-defined problems, especially insight problems, have been studied for a long time (but not as widely)
  - Some key concepts are well established, but very difficult to develop general theories or models for the way people solve these problems
Practice Question

• In the study of human problem solving, a well defined problem is one that involves
  a) An initial or starting state
  b) A clear end or goal state
  c) A set of allowable moves or actions to change states
  d) All of the above
Practice Question

- A business person needs to take a taxi cab from their hotel to the office in which they are working every day for a 5 day visit. They have a choice of 3 taxi companies, and, because they have a company expense account for the fares, just want as many of their rides as possible to be enjoyable. The problem the business person is solving in choosing taxi companies is most similar to:
  a) the towers of Hanoi problem
  b) a traveling salesperson problem
  c) an optimal stopping problem
  d) a bandit problem
Practice Question

- Which of the following problems requires people to deal with uncertainty in developing their solution, rather than just a potentially large state space to search?
  
  a) A minimum Spanning tree problem.
  b) An optimal stopping problem.
  c) A visual maze problem.
  d) A cannibals and missionaries problem.
Practice Question

- List three heuristics that have been proposed as part of the way people guide their search in solving problems. Just name the heuristics. You do not need to describe them.