Math 112C final exam practice problems with solutions

Please note that this document only contains problems from the 2nd half of the course. Problems from the 1st half of the course will also be included in the final. Please use the practice midterm to study for that part.

1. Consider the eigenproblem:

\[(xu')' - \frac{m^2}{x}u + \lambda xu = 0, \quad 0 < x < 1,\]
\[u(1) = 0, \quad 0 \leq x \leq 1.\]

What are the eigenvalues and eigenfunctions of this problem? Expand \(f(x)\) in terms of these eigenfunctions (write down the general formulas for the infinite series and the coefficients).

**Solution:**

We know that the solutions to this problem are the eigenfunctions \(J_m \left( \sqrt{\lambda_{k}^{(m)}} x \right)\), where \(J_m\) is the Bessel function of order \(m\) given by

\[J_m(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} t \right)^{m+2k}}{k!(m+k)!}.\]

**You do not have to remember this formula!** The eigenvalues are found by the roots of the equation

\[J_m \left( \sqrt{\lambda} \right) = 0.\]

The function \(f\) can be expanded using

\[f(x) = \sum_{k=1}^{\infty} c_k J_m \left( \sqrt{\lambda_k^{(m)}} x \right),\]

where

\[c_k = \frac{\int_0^1 x f(x) J_m \left( \sqrt{\lambda_k^{(m)}} x \right) \, dx}{\int_0^1 x \left( J_m \left( \sqrt{\lambda_k^{(m)}} x \right) \right)^2 \, dx}.\]
2. Find a series solution to the problem for a circular membrane,

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) = 0, \quad r < 1, \quad t > 0, \tag{4}
\]

\[u(1, \theta, t) = 0, \tag{5}\]

\[u(r, \theta, 0) = 0, \tag{6}\]

\[\frac{\partial u(r, \theta, 0)}{\partial t} = g(r, \theta). \tag{7}\]

**Solution:**

This equation is solved using separation of variables. We assume that \(u(r, \theta, t) = R(r)\Theta(\theta)T(t)\). Substitution into the equation gives

\[R\Theta T'' - c^2 \left( R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' \right) = 0,
\]

which can be written as

\[\frac{T''}{c^2 T} = \frac{R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta''}{R\Theta}.
\]

Since the left-hand side is a function of \(t\) only, and the right hand side is a function of \(r\) and \(\theta\) only, we conclude that both sides must be equal to a constant. Thus we have

\[\frac{T''}{c^2 T} = \frac{R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta''}{R\Theta} = -\lambda.
\]

Considering the equation for \(T\), we get

\[T'' + c^2 \lambda T = 0,
\]

with one initial condition taken from the original problem:

\[T(0) = 0.
\]

The solution to the initial value problem is given by

\[T(t) = \sin \left( c\sqrt{\lambda} t \right).\]
Now we take the equation with $R$ and $\Theta$ and separate variables. The equation becomes

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = -\lambda R\Theta,$$

which is written as

$$\frac{\Theta''}{\Theta} = \frac{R'' + \frac{1}{r}R' + \lambda R}{-\frac{1}{r^2}R} = -m^2.$$

Again, since the two sides are functions of different variables, we assume both sides are constant. We have

$$\frac{\Theta''}{\Theta} = \frac{R'' + \frac{1}{r}R' + \lambda R}{-\frac{1}{r^2}R} = -m^2.$$

The equation for $\Theta$ is

$$\Theta'' + m^2\Theta = 0,$$

together with the continuity conditions,

$$\Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi).$$

This is an eigenvalue problem which has nontrivial solutions for integer values of $m$. The corresponding solutions are:

$$\Theta(\theta) = \sin(m\theta), \quad \Theta(\theta) = \cos(m\theta).$$

The equation for $R$ is

$$R'' + \frac{1}{r}R' + \lambda R = \frac{m^2}{r^2} R,$$

or

$$(rR')' - \frac{m^2}{r} R + \lambda rR = 0,$$

with the boundary condition

$$R(1) = 0,$$

and the requirement of boundedness. The solution for a particular $m$ is known to be given by the Bessel functions. We also know that Bessel’s equation is an eigenvalue problem, with a discrete set of eigenvalues for each given $m$: $\lambda_1^{(m)}$, $\lambda_2^{(m)}$, etc. The eigenfunctions are given by

$$R^{(m)}_k(r) = J_m \left( \sqrt{\lambda_k^{(m)}} r \right).$$
Putting everything together in a series solution and making sure the non-homogeneous boundary condition $\frac{\partial u(r, \theta, t)}{\partial t} = g(r, \theta)$ is satisfied, we obtain:

$$u(r, \theta, t) = \frac{1}{2} \sum_{k=1}^{\infty} c_{k0} J_0 \left( \sqrt{\lambda_k^{(0)}} r \right) \frac{\sin \left( c \sqrt{\lambda_k^{(0)}} t \right)}{c \sqrt{\lambda_k^{(0)}}},$$

$$+ \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (c_{km} \cos(m\theta) + d_{km} \sin(m\theta)) J_m \left( \sqrt{\lambda_k^{(m)}} r \right) \frac{\sin \left( c \sqrt{\lambda_k^{(m)}} t \right)}{c \sqrt{\lambda_k^{(m)}}},$$

where

$$c_{km} = \frac{\int_{-\pi}^{\pi} \int_{0}^{1} g(r, \theta) \cos(m\theta) J_m \left( \sqrt{\lambda_k^{(m)}} r \right) r \, dr \, d\theta}{\pi \int_{0}^{1} \left( J_m \left( \sqrt{\lambda_k^{(m)}} r \right) \right)^2 r \, dr},$$

and

$$d_{km} = \frac{\int_{-\pi}^{\pi} \int_{0}^{1} g(r, \theta) \sin(m\theta) J_m \left( \sqrt{\lambda_k^{(m)}} r \right) r \, dr \, d\theta}{\pi \int_{0}^{1} \left( J_m \left( \sqrt{\lambda_k^{(m)}} r \right) \right)^2 r \, dr}.$$
3. Solve the ODE describing forced oscillations of a pendulum:

\[
\frac{d^2 y}{dt^2} + W^2 y = L \sin(\omega t),
\]

(8)

\[
y(0) = 0, \quad \frac{dy}{dt}(0) = 0,
\]

(9)

where \( y(t) \) describes the displacement of the pendulum from the vertical.

What happens when \( \omega \to W \) (describe the phenomenon of resonance)? Use the trigonometric facts:

\[
\cos u - \cos v = 2 \sin \frac{u + v}{2} \sin \frac{v - u}{2},
\]

(10)

\[
\sin u - \sin v = 2 \cos \frac{u + v}{2} \sin \frac{u - v}{2}.
\]

(11)

**Solution:**

We first consider the homogeneous equation

\[
y'' + W^2 y = 0,
\]

where the general solution is clearly

\[
y(t) = c_1 \sin(Wt) + c_2 \cos(Wt).
\]

We know the solution to the nonhomogeneous equation is of the form

\[
y(t) = c_1 \sin(Wt) + c_2 \cos(Wt) + \psi(t),
\]

where \( \psi(t) \) is a particular solution to the nonhomogeneous equation. We make a judicious guess that \( \psi(t) = A \sin(\omega t) \). Substitution into the equation gives

\[
-A\omega^2 \sin(\omega t) + W^2 A \sin(\omega t) = L \sin(\omega t).
\]

From this we know that

\[
-A\omega^2 + W^2 A = L,
\]

or

\[
A = \frac{L}{W^2 - \omega^2}.
\]
So the general solution to the nonhomogeneous equation is

\[ y(t) = c_1 \sin(Wt) + c_2 \cos(Wt) + \frac{L}{W^2 - \omega^2} \sin(\omega t). \]

Now the initial conditions are used to obtain the coefficients. The condition \( y(0) = 0 \) implies that \( c_2 = 0 \), so we now have a solution of the form

\[ y(t) = c_1 \sin(Wt) + \frac{L}{W^2 - \omega^2} \sin(\omega t). \]

Calculating the derivative gives

\[ y'(t) = c_1 W \cos(Wt) + \frac{\omega L}{W^2 - \omega^2} \cos(\omega t). \]

Using the initial condition \( y'(0) = 0 \) here gives

\[ c_1 W + \frac{\omega L}{W^2 - \omega^2} = 0, \]

or

\[ c_1 = \frac{\omega L}{W(\omega^2 - W^2)}. \]

Thus the solution to the problem is

\[ y(t) = \frac{\omega L}{W(\omega^2 - W^2)} \sin(Wt) - \frac{L}{\omega^2 - W^2} \sin(\omega t). \]

Now we examine the solution as \( \omega \to W \). We write the solution as

\[ y(t) = \frac{L}{W(\omega^2 - W^2)} (\omega \sin(Wt) - W \sin(\omega t)), \]

then we add and subtract \( \omega \sin(\omega t) \) inside the parentheses, so that

\[ y(t) = \frac{L}{W(\omega^2 - W^2)} (\omega \sin(Wt) - \omega \sin(\omega t) + \omega \sin(\omega t) - W \sin(\omega t)). \]

Using the formula given in the hint we have

\[ y(t) = \frac{L}{W(\omega^2 - W^2)} \left( 2\omega \cos \left( \frac{W + \omega}{2} t \right) \sin \left( \frac{W - \omega}{2} t \right) + (\omega \sin(\omega t) - W \sin(\omega t)) \right). \]
We continue rewriting

\[
y(t) = \frac{2\omega L}{W(\omega + W)(\omega - W)} \cos \left( \frac{W + \omega}{2} t \right) \sin \left( \frac{W - \omega}{2} t \right) + \frac{L(\omega - W)}{W(\omega + W)(\omega - W)} \sin(\omega t),
\]

or

\[
y(t) = -\frac{2\omega L}{W(W + \omega)} \cos \left( \frac{W + \omega}{2} t \right) \cdot \frac{1}{W - \omega} \sin \left( \frac{W - \omega}{2} t \right) + \frac{L}{W(W + W)} \sin(\omega t).
\]

We know that

\[
\lim_{x \to 0} \frac{\sin(Ax)}{x} = A,
\]

therefore as \( \omega \to W \) we see that

\[
y(t) = -\frac{L}{W} \cos(Wt) \cdot \frac{t}{2} + \frac{L}{2W^2} \sin(Wt),
\]

or

\[
y(t) = \frac{L}{2W^2} \left( \sin(Wt) - Wt \cos(Wt) \right).
\]

This function is unbounded as \( t \to \infty \).
4. Consider a forced pendulum with friction:

\[
\frac{d^2y}{dt^2} + k \frac{dy}{dt} + W^2 y = L \sin(\omega t),
\]

\[
y(0) = 0, \quad \frac{dy}{dt}(0) = 0.
\]

Find the solution \(y(t)\) and describe what happens if \(\omega \to W\).

**Solution:**

We first consider the homogeneous equation

\[y'' + ky' + W^2 y = 0.\]

To obtain the general solution we look at the characteristic equation

\[r^2 + kr + W^2 = 0,\]

from which we see that

\[r = \frac{-k \pm \sqrt{k^2 - 4W^2}}{2}.\]

The case \(k^2 \geq 4W^2\) corresponds to the existence of very strong friction, such that a free pendulum does not oscillate but rather stops after one swing. We do not consider this case here. Assuming that \(k^2 < 4W^2\) we have that

\[r = \frac{-k \pm \sqrt{4W^2 - k^2}}{2}i,\]

from which we obtain the general solution

\[y(t) = c_1 e^{-\frac{k}{2}t} \sin \left(\frac{1}{2} \sqrt{4W^2 - k^2} \ t \right) + c_2 e^{-\frac{k}{2}t} \cos \left(\frac{1}{2} \sqrt{4W^2 - k^2} \ t \right).\]

We know the solution to the nonhomogeneous equation is of the form

\[y(t) = c_1 e^{-\frac{k}{2}t} \sin \left(\frac{1}{2} \sqrt{4W^2 - k^2} \ t \right) + c_2 e^{-\frac{k}{2}t} \cos \left(\frac{1}{2} \sqrt{4W^2 - k^2} \ t \right) + \psi(t),\]
where $\psi(t)$ is a particular solution to the nonhomogeneous equation. We make a judicious guess that $\psi(t) = A\sin(\omega t) + B\cos(\omega t)$. Substitution into the equation gives

$$
-A\omega^2 \sin(\omega t) - B\omega^2 \cos(\omega t) \\
+kA\omega \cos(\omega t) - kB\omega \sin(\omega t) \\
+W^2 A \sin(\omega t) + W^2 B \cos(\omega t) = L \sin(\omega t).
$$

From this we know that

$$
-A\omega^2 - kB\omega + W^2 A = L,
$$

and

$$
-B\omega^2 + kA\omega + W^2 B = 0.
$$

The first equation gives

$$
A = \frac{L + kB\omega}{W^2 - \omega^2},
$$

and substitution back into the second equation gives

$$
(W^2 - \omega^2) B + k\omega \frac{L + kB\omega}{W^2 - \omega^2} = 0,
$$

which can be solved for $B$ giving

$$
B = -\frac{k\omega L}{(W^2 - \omega^2)^2 + k^2\omega^2}.
$$

Then we can find $A$ as

$$
A = \frac{(W^2 - \omega^2) L}{(W^2 - \omega^2)^2 + k^2\omega^2}.
$$

So the general solution to the nonhomogeneous equation is

$$
y(t) = c_1 e^{-\frac{k}{2}t} \sin \left( \frac{1}{2} \sqrt{4W^2 - k^2} \ t \right) + c_2 e^{-\frac{k}{2}t} \cos \left( \frac{1}{2} \sqrt{4W^2 - k^2} \ t \right) \\
+ \frac{(W^2 - \omega^2) L}{(W^2 - \omega^2)^2 + k^2\omega^2} \sin(\omega t) - \frac{k\omega L}{(W^2 - \omega^2)^2 + k^2\omega^2} \cos(\omega t)
$$

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Now the initial conditions are used to obtain the coefficients. The condition $y(0) = 0$ implies that
\[
c_2 = \frac{k\omega L}{(W^2 - \omega^2)^2 + k^2\omega^2}.
\]
Calculating the derivative of $y$ gives
\[
y'(t) = c_1 \left( -\frac{k}{2} e^{-\frac{t}{2}} \sin \left( \frac{1}{2} \sqrt{4W^2 - k^2} t \right) + \frac{1}{2} \sqrt{4W^2 - k^2} e^{-\frac{t}{2}} \cos \left( \frac{1}{2} \sqrt{4W^2 - k^2} t \right) \right)
\]
\[+ c_2 \left( -\frac{k}{2} e^{-\frac{t}{2}} \cos \left( \frac{1}{2} \sqrt{4W^2 - k^2} t \right) - \frac{1}{2} \sqrt{4W^2 - k^2} e^{-\frac{t}{2}} \sin \left( \frac{1}{2} \sqrt{4W^2 - k^2} t \right) \right)
\]
\[+ \frac{\omega (W^2 - \omega^2) L}{(W^2 - \omega^2)^2 + k^2\omega^2} \cos(\omega t) + \frac{k\omega L}{(W^2 - \omega^2)^2 + k^2\omega^2} \sin(\omega t).
\]
Using the initial condition $y'(0) = 0$ here gives
\[
\frac{c_1}{2} \sqrt{4W^2 - k^2} - \frac{c_2 k}{2} + \frac{\omega (W^2 - \omega^2) L}{(W^2 - \omega^2)^2 + k^2\omega^2} = 0.
\]
Substituting $c_2$ from above and solving for $c_1$ gives
\[
c_1 = \frac{k^2\omega L - 2\omega (W^2 - \omega^2) L}{\sqrt{4W^2 - k^2} ((W^2 - \omega^2)^2 + k^2\omega^2)}.
\]
Thus the solution to the problem is
\[
y(t) = \frac{k^2\omega L - 2\omega (W^2 - \omega^2) L}{\sqrt{4W^2 - k^2} ((W^2 - \omega^2)^2 + k^2\omega^2)} e^{-\frac{t}{2}} \sin \left( \frac{1}{2} \sqrt{4W^2 - k^2} t \right)
\]
\[+ \frac{k\omega L}{(W^2 - \omega^2)^2 + k^2\omega^2} e^{-\frac{t}{2}} \cos \left( \frac{1}{2} \sqrt{4W^2 - k^2} t \right)
\]
\[+ \frac{(W^2 - \omega^2) L}{(W^2 - \omega^2)^2 + k^2\omega^2} \cos(\omega t) - \frac{k\omega L}{(W^2 - \omega^2)^2 + k^2\omega^2} \sin(\omega t)
\]
If we examine the solution as $\omega \to W$ we see that
\[
y(t) = \frac{L}{W\sqrt{4W^2 - k^2}} e^{-\frac{t}{2}} \sin \left( \frac{1}{2} \sqrt{4W^2 - k^2} t \right)
\]
\[+ \frac{L}{kW} e^{-\frac{t}{2}} \cos \left( \frac{1}{2} \sqrt{4W^2 - k^2} t \right) - \frac{L}{kW} \cos(Wt)
\]
In the case that $k > 0$ we see that the first two terms will decay as $t \to \infty$, so that $y$ will look like

$$y(t) \approx -\frac{L}{kW} \cos(Wt)$$

as $t$ increases.
5. Consider the problem for forced vibrations of a square membrane:

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = F(x, y) \cos(\omega t), \quad 1 < x, y < 5, \quad t > 0, \quad (14) \]

\[ u(1, y, t) = u(5, y, t) = u(x, 1, t) = u(x, 5, t) = 0, \quad (15) \]

\[ u(x, y, 0) = \frac{\partial u}{\partial t}(x, y, 0) = 0. \quad (16) \]

Make a change of variables to reduce this to a problem in a square 0 < x′, y′ < π and solve the problem. Write down the solution in the original variables.

**Solution:**

We consider a change of variables given by

\[ x' = \frac{\pi}{4} (x - 1), \quad y' = \frac{\pi}{4} (y - 1). \]

We need to calculate \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) in terms of x′ and y′. First we use the chain rule to calculate first derivatives, so

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial x}. \]

A simple calculation shows that \( \frac{\partial x'}{\partial x} = \frac{\pi}{4} \) and \( \frac{\partial y'}{\partial x} = 0 \), so

\[ \frac{\partial u}{\partial x} = \frac{\pi u}{4 \partial x'}. \]

Calculating the second derivative gives

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\pi}{4} \left( \frac{\partial^2 u}{\partial x'^2} \frac{\partial x'}{\partial x} + \frac{\partial^2 u}{\partial x' \partial y'} \frac{\partial y'}{\partial x} \right), \]

which simplifies to

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\pi^2}{16} \frac{\partial^2 u}{\partial x'^2}. \]

Similarly we can show that

\[ \frac{\partial^2 u}{\partial y^2} = \frac{\pi^2}{16} \frac{\partial^2 u}{\partial y'^2}. \]
Therefore the equation becomes
\[
\frac{\partial^2 u}{\partial t^2} - \frac{\pi^2 c^2}{16} \left( \frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} \right) = F \left( \frac{4}{\pi} x' + 1, \frac{4}{\pi} y' + 1 \right) \cos(\omega t), \quad 0 < x', y' < \pi, \quad t > 0,
\]
\[
u(0, y', t) = u(\pi, y', t) = u(x', 0, t) = u(x', \pi, t) = 0,
\]
\[
u(x', y', 0) = \frac{\partial u}{\partial t}(x', y', 0) = 0.
\]
The equations
\[
X'' + m^2 X = 0, \quad Y'' + n^2 Y = 0,
\]
along with the boundary conditions for the problem lead to the solutions
\[
X(x') = \sin(mx'), \quad Y(y') = \sin(ny').
\]
From this we know that \( u \) may be represented as a series of the form
\[
u(x', y', t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}(t) \sin(mx') \sin(ny').
\]
We also represent \( \bar{F}(x', y') = F \left( \frac{4}{\pi} x' + 1, \frac{4}{\pi} y' + 1 \right) \) as a series
\[
\bar{F}(x', y') = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin(mx') \sin(ny'),
\]
where
\[
a_{mn} = \frac{4}{\pi^2} \int_{0}^{\pi} \int_{0}^{\pi} \bar{F}(x', y') \sin(mx') \sin(ny') \, dx' dy'.
\]
Substitution into the differential equation gives
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}'(t) \sin(mx') \sin(ny')
\]
\[
+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi^2 c^2 m^2}{16} c_{mn}(t) \sin(mx') \sin(ny')
\]
\[
+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi^2 c^2 n^2}{16} c_{mn}(t) \sin(mx') \sin(ny')
\]
\[
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \cos(\omega t) \sin(mx') \sin(ny').
\]
From this we know that
\[ c''_{mn}(t) + \frac{\pi^2 c^2 (m^2 + n^2)}{16} c_{mn}(t) = a_{mn} \cos(\omega t). \]

The solution to the homogeneous equation is
\[ c_{mn}(t) = c_1 \sin \left( \frac{\pi c \sqrt{m^2 + n^2}}{4} t \right) + c_2 \cos \left( \frac{\pi c \sqrt{m^2 + n^2}}{4} t \right). \]

The solution to the nonhomogeneous equation is of the form
\[ c_{mn}(t) = c_1 \sin \left( \frac{\pi c \sqrt{m^2 + n^2}}{4} t \right) + c_2 \cos \left( \frac{\pi c \sqrt{m^2 + n^2}}{4} t \right) + \psi(t). \]

We make a judicious guess that \( \psi(t) = A \cos(\omega t) \), and substitute into the equation giving
\[ -A \omega^2 \cos(\omega t) + \frac{\pi^2 c^2 (m^2 + n^2)}{16} A \cos(\omega t) = a_{mn} \cos(\omega t). \]

This implies that
\[ -A \omega^2 + \frac{\pi^2 c^2 (m^2 + n^2)}{16} A = a_{mn}, \]
so
\[ A = -\frac{16a_{mn}}{16\omega^2 - \pi^2 c^2 (m^2 + n^2)}. \]

Thus the general solution to the ordinary differential equation is
\[ c_{mn}(t) = c_1 \sin \left( \frac{\pi c \sqrt{m^2 + n^2}}{4} t \right) + c_2 \cos \left( \frac{\pi c \sqrt{m^2 + n^2}}{4} t \right) - \frac{16a_{mn}}{16\omega^2 - \pi^2 c^2 (m^2 + n^2)} \cos(\omega t). \]

Now we take the initial conditions from the original problem and find the coefficients. We have
\[ c_{mn}(0) = 0, \quad c'_{mn}(0) = 0. \]

The initial condition that \( c_{mn}(0) = 0 \) implies that
\[ c_2 = \frac{16a_{mn}}{16\omega^2 - \pi^2 c^2 (m^2 + n^2)}. \]
and the initial condition that $c'_mn(0) = 0$ implies that

$$c_1 = 0.$$ 

So the solution is

$$c_{mn}(t) = \frac{16a_{mn}}{16\omega^2 - \pi^2c^2(m^2 + n^2)} \left( \cos \left( \frac{\pi c\sqrt{m^2 + n^2}}{4} t \right) - \cos(\omega t) \right).$$

Now we find the series solution for $u$ of the form

$$u(x', y', t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16a_{mn}}{16\omega^2 - \pi^2c^2(m^2 + n^2)} \left( \cos \left( \frac{\pi c\sqrt{m^2 + n^2}}{4} t \right) - \cos(\omega t) \right) \sin(mx') \sin(ny'),$$

with

$$a_{mn} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} \bar{F}(x', y') \sin(mx') \sin(ny') \ dx' dy'.$$

Now we switch back to the original coordinate system giving

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16a_{mn}}{16\omega^2 - \pi^2c^2(m^2 + n^2)} \left( \cos \left( \frac{\pi c\sqrt{m^2 + n^2}}{4} t \right) - \cos(\omega t) \right) \cdot \sin \left( \frac{m\pi}{4} (x - 1) \right) \sin \left( \frac{n\pi}{4} (y - 1) \right).$$

The coefficients are also converted to the original coordinates so that

$$a_{mn} = \frac{4}{\pi^2} \int_1^{5} \int_1^{5} \bar{F} \left( \frac{\pi}{4} (x - 1), \frac{\pi}{4} (y - 1) \right) \sin \left( \frac{m\pi}{4} (x - 1) \right) \sin \left( \frac{n\pi}{4} (y - 1) \right) \frac{\pi^2}{16} \ dx \ dy,$$

or

$$a_{mn} = \frac{1}{4} \int_1^{5} \int_1^{5} F(x, y) \sin \left( \frac{m\pi}{4} (x - 1) \right) \sin \left( \frac{n\pi}{4} (y - 1) \right) \ dx \ dy,$$
6. Recall facts about Legendre polynomials and Legendre associated functions. (a) Calculate \( \int_0^\pi P_{15}(\cos \theta) \sin \theta \, d\theta \). (b) Calculate \( \int_{-1}^1 P_7^7(t)P_6^7(t) \, dt \). (c) Find \( P_3(t) \) if \( P_0(t) = 1, \, P_1(t) = t, \, P_2(t) = 3t^2/2 - 1/2 \), by using the orthogonality and normalization properties. (d) How many roots does the function \( P_{23}(t) \) have between \((-1)\) and \(1)\?

**Solution:**

(a) Using the substitution \( t = \cos \theta \) we have

\[
\int_0^\pi P_{15}(\cos \theta) \sin \theta \, d\theta = \int_{-1}^1 P_{15}(t) \, dt = 0.
\]

We know the answer is zero because \( P_{15}(t) \) is orthogonal to \( P_0(t) = 1 \).

(b) We know that

\[
P_7^4(t) = (1 - t^2)^{7/2} \frac{d^7}{dt^7} [P_4(t)].
\]

Since \( P_4(t) \) is a polynomial of degree 4, we know that

\[
\frac{d^7}{dt^7} [P_4(t)] = 0,
\]

so

\[
P_7^4(t) = 0.
\]

Similarly

\[
P_7^6(t) = 0.
\]

Therefore

\[
\int_{-1}^1 P_7^4(t)P_6^7(t) \, dt = 0.
\]

(In general, we must have \( n \geq m \), otherwise the Associated Legendre function is zero). **Remark:** In general, if the index \( m \) is the same for the two polynomials, and \( n \) is different, then they are orthogonal to each other.

(c) We assume that \( P_3(t) = at^3 + bt^2 + ct + d \). The normalization property is that \( P_3(1) = 1 \), so we have

\[
a + b + c + d = 1.
\]
Orthogonality with $P_0(t)$ gives $\int_{-1}^{1} P_0(t)P_3(t)\ dt = 0$, so
\[
\int_{-1}^{1} (at^3 + bt^2 + ct + d) \ dt = 0
\]
\[
\frac{at^4}{4} + \frac{bt^3}{3} + \frac{ct^2}{2} + dt|_{-1}^{1} = 0
\]
\[
\frac{2b}{3} + 2d = 0
\]

Orthogonality with $P_1(t)$ gives $\int_{-1}^{1} P_1(t)P_3(t)\ dt = 0$, so
\[
\int_{-1}^{1} t(at^3 + bt^2 + ct + d) \ dt = 0
\]
\[
\int_{-1}^{1} (at^4 + bt^3 + ct^2 + dt) \ dt = 0
\]
\[
\frac{at^5}{5} + \frac{bt^4}{4} + \frac{ct^3}{3} + dt^2|_{-1}^{1} = 0
\]
\[
\frac{2a}{5} + \frac{2c}{3} = 0
\]

Orthogonality with $P_2(t)$ gives $\int_{-1}^{1} P_2(t)P_3(t)\ dt = 0$, so
\[
\int_{-1}^{1} \left( \frac{3t^2}{2} - \frac{1}{2} \right) (at^3 + bt^2 + ct + d) \ dt = 0
\]
\[
\int_{-1}^{1} \left( \frac{3at^5}{2} + \frac{3bt^4}{2} + \frac{3ct^3}{2} + \frac{3dt^2}{2} - \frac{at^3}{2} - \frac{bt^2}{2} - \frac{ct}{2} - \frac{d}{2} \right) \ dt = 0
\]
\[
\int_{-1}^{1} \left( \frac{3at^5}{2} + \frac{3bt^4}{2} + \frac{(3c-a)t^3}{2} + \frac{(3d-b)t^2}{2} - \frac{ct}{2} - \frac{d}{2} \right) \ dt = 0
\]
\[
\frac{at^6}{4} + \frac{3bt^5}{10} + \frac{(3c-a)t^4}{8} + \frac{(3d-b)t^3}{6} - \frac{ct^2}{4} - \frac{dt^2}{2}|_{-1}^{1} = 0
\]
\[
\frac{3b}{5} + \frac{3d-b}{3} - d = 0
\]
\[
\frac{4b}{15} = 0
\]
\[
b = 0
\]

The result that $b = 0$ combined with $\frac{2b}{3} + 2d = 0$ implies that $d = 0$. Then we have a system of 2 linear equations in two unknowns
\[
a + c = 1
\]
\[
\frac{2a}{5} + \frac{2c}{3} = 0
\]

The solution to this system is

\[
a = \frac{5}{2}, \quad c = -\frac{3}{2}.
\]

So we conclude that

\[
P_3(t) = \frac{5t^3}{2} - \frac{3t}{2}.
\]

(d) It is known that \(P_n(t)\) has \(n\) roots between -1 and 1. Therefore \(P_{23}(t)\) has 23 roots between -1 and 1.
7. Solve the Laplace equation in a sphere:

\[
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0, \quad r < R, \quad (17)
\]

\[u(R, \theta, \phi) = \sin(3\phi). \quad (18)\]

In your solution, use the fact that

\[
\int_0^\pi P_n^m(\cos \theta)^2 \sin \theta \, d\theta = \frac{2}{2n + 1} (n + m)!,
\]

and denote

\[
\int_0^\pi P_n^m(\cos \theta) \sin \theta \, d\theta = K_n^m.
\]

What is the value \(u(0, \phi, \theta)\)?

**Solution:**

The solution to this problem is found by expanding \(f(\theta, \phi) = \sin(3\phi)\) in a double Fourier series

\[
f(\theta, \phi) = \sum_{n=0}^\infty \left[ \frac{1}{2} a_{n0} P_n(\cos \theta) + \sum_{m=1}^n \left( a_{nm} \cos(m\phi) + b_{nm} \sin(m\phi) \right) P_n^m(\cos \theta) \right].
\]

Clearly we must have \(a_{nm} = 0\) for all \(m, n \geq 0\) and \(b_{nm} = 0\) when \(m \neq 3\). So we have

\[
\sin(3\phi) = \sum_{n=3}^\infty b_{n3} P_n^3(\cos \theta) \sin(3\phi).
\]

Since

\[
b_{nm} = \frac{\int_0^{2\pi} \int_0^\pi f(\theta, \phi) P_n^m(\cos \theta) \sin(m\phi) \sin \theta \, d\theta d\phi}{\pi \int_0^\pi P_n^m(\cos \theta)^2 \sin \theta \, d\theta},
\]

we know that

\[
b_{n3} = \frac{\int_0^{2\pi} \int_0^\pi \sin(3\phi) P_n^3(\cos \theta) \sin(3\phi) \sin \theta \, d\theta d\phi}{\pi \int_0^\pi P_n^3(\cos \theta)^2 \sin \theta \, d\theta}.
\]

We need to calculate \(\int_0^{2\pi} \sin^2(3\phi) \, d\phi\), so

\[
\int_0^{2\pi} \sin^2(3\phi) \, d\phi = \int_0^{2\pi} \frac{1 - \cos(6\phi)}{2} \, d\phi
\]

\[
= \frac{\phi}{2} - \frac{\sin(6\phi)}{12} \bigg|_0^{2\pi}
\]

\[
= \pi.
\]
Therefore

\[ b_{n3} = \int_0^\pi P^3_n(\cos \theta) \sin \theta \, d\theta \int_0^\pi P^3_n(\cos \theta)^2 \sin \theta \, d\theta, \]

or

\[ b_{n3} = \frac{2n + 1}{2} \frac{(n - 3)!}{(n + 3)!} K^3_n. \]

The solution is

\[ u(r, \theta, \phi) = \sum_{n=3}^{\infty} \left( \frac{r}{R} \right)^n b_{n3} \sin(3\phi) P^3_n(\cos \theta), \]

with \( b_{n3} \) defined as above. Apparently \( u(0, \theta, \phi) = 0 \). Remark: One could also use the Mean Value theorem to calculate the value of \( u \) in the middle. It is given by

\[ u(0, \theta, \phi) = \frac{1}{4\pi} \int_0^\pi \int_{-\pi}^{\pi} f(\theta, \phi) \sin \theta \, d\theta \, d\phi. \]

8. Consider the inviscid Burgers' equation with the initial condition \( u(x,0) = u_0(x) \). (a) Find the solution \( u(x,t) \) by using the explicit formula, if \( u_0(x) = x/2 + 1 \). (b) Find equations for the characteristics and draw the characteristics in the \((x,t)\) plane, for \( u_0 = 1 \) with \( x < 0 \), \( u_0 = 1 - x \), \( 0 \leq x \leq 1 \), and \( u_0 = 0 \) with \( x > 1 \). (c) Will a shock form in the case with \( u_0(x) \) as in (b)? If so, find the earliest time of shock formation.

Solution:

(a) We have \( u = u_0(x - ut) \), to be resolved for the unknown \( u \). Using \( u_0(x) = x + 1 \) we have \( u = (x - ut)/2 + 1 \), which gives \( u(x,t) = \frac{x^2}{t^2} \times \frac{1}{2} + 1 \). (b) The characteristics have equations \( x = t + c \) for \( c < 0 \), \( x = (1 - c)t + c \) with \( 0 \leq c \leq 1 \), and \( x = c \) (horizontal lines) with \( c > 1 \). Note that all the characteristics with \( 0 \leq c \leq 1 \) cross at the point \((1,1)\). (c) Since the initial condition \( u_0(x) \) has a negative slope, a shock will form. The earliest time of shock formation is \( t_* = \min_x \left( -\frac{1}{u_0'(x)} \right) \), for all \( x \) where \( u'(x) < 0 \). We have \( u_0' = -1 \), therefore \( t_* = 1 \).