Math 112A final exam practice problems

ATTENTION: The final exam will contain problems from the first and the second parts of the course. Here I only present problems from the second half of the course (after the midterm). Use the midterm preparation materials as well as this sheet to prepare for the final. Most problems will be from the second half of the course. I included solutions to most of the problems.

1. Consider a wave equation on an infinite line,
\[ \frac{\partial^2 u}{\partial t^2} - \frac{1}{x^4} \frac{\partial^2 u}{\partial x^2} = 0. \]
Find the characteristics though the point (0, 3). Draw the domains of dependence and influence of the point (0, 3) (for \( t \geq 0 \)).

**Solution:** The equations of characteristics are \( t = \pm \frac{x^3}{3} + 3 \). The domain of dependence is below the two curves. The domain of influence is above.

2. Is the equation
\[ \frac{\partial^2 u}{\partial t^2} + 4 \frac{\partial^2 u}{\partial x \partial t} - \frac{9}{4} \frac{\partial^2 u}{\partial x^2} = 0 \]
hyperbolic, elliptic or parabolic (explain)? Find the general equations for characteristics if possible.

**Solution:** The equation is hyperbolic because \( A^2 - 4BC = 25 > 0 \). The characteristics are \( \xi = 2x + t, \eta = 2x - 9t \).

3. Consider the Laplace equation,
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]
defined in the inside of the circle, \( x^2 + y^2 = 25 \). The boundary condition is given by \( u(x, y) = x \) for such \((x, y)\) that \( x^2 + y^2 = 25 \). Can we have \( u(x, y) = 10 \) somewhere inside the circle? (Hint: use the maximum principle, p. 56.)

**Solution.** According to the maximum principle, the value of the function \( u(x, t) \) in the whole domain must be contained between \( m \) and \( M \), where \( m \) is the minimum value on the boundary and \( M \) is the maximum value on the
boundary. For this problem, the max value on the boundary is 5. To see
this, draw the domain (a circle centered at zero with radius 5) ans see that
the max value of \( x \) is 5. Therefore, the value of \( u(x, y) \) in the interior cannot
exceed 5, and thus it cannot be equal to 10.

4. Consider functions \( \phi_1(x) = \sin x \) and \( \phi_2(x) = \cos x \) on the interval
\([0, \pi]\)? (a) Are they orthogonal on the interval \([-\pi, \pi]\)? Take \( \rho(x) = 1 \).
Prove. (b) What is the best approximation to function \( f(x) = x \) in terms of
the functions \( \phi_1(x) \) and \( \phi_2(x) \).

Solution. (a) Two functions are orthogonal if \( \int_{-\pi}^{\pi} \phi_1(x)\phi_2(x)\rho(x) \, dx = 0 \).
We have \( \int_{-\pi}^{\pi} \sin x \cos s \, dx = 2 \int_{0}^{\pi} \sin(2x) \, dx = 0 \), and \( \int_{-\pi}^{\pi} \sin x \cos s \, dx = 2 \int_{-\pi}^{\pi} \sin(2x) \, dx = 0 \), so they are orthogonal on both intervals. (b) For each
coefficient,

\[
c_i = \frac{\int_{-\pi}^{\pi} f(x)\phi_i(x)\rho(x) \, dx}{\int_{-\pi}^{\pi} \phi_i^2(x)\rho(x) \, dx}
\]

which gives

\[
c_1 = \frac{\int_{-\pi}^{\pi} x \sin(x) \, dx}{\int_{-\pi}^{\pi} \sin^2(x) \, dx} = -x \cos(x)|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos(x) \, dx
\]

\[
= \frac{-\pi(-1) + (-\pi)(-1) + \sin(x)|_{-\pi}^{\pi}}{\left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right)|_{-\pi}^{\pi} = \frac{2\pi}{\pi} = 2}
\]

and

\[
c_1 = \frac{\int_{-\pi}^{\pi} x \cos(x) \, dx}{\int_{-\pi}^{\pi} \cos^2(x) \, dx} = x \sin(x)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin(x) \, dx
\]

\[
= \frac{0 - \cos(x)|_{-\pi}^{\pi}}{\left(\frac{1}{2}x + \frac{1}{4}\sin(2x)\right)|_{-\pi}^{\pi} = 0\pi = 0}
\]

and the best approximation is \( s_2 = 2 \sin(x) \).

5. Consider the initial-boundary value problem:

\[
\frac{\partial u}{\partial t} - 2\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \pi, \quad (1)
\]

\[
u(0, t) = u(\pi, t) = 0, \quad (2)
\]

\[
u(x, 0) = f(x) = \sum_{n=1}^{\infty} \frac{\sin n x}{n!}.
\]

(3)
Solve the problem by the method of separation of variables. (a) Present \( u(x,t) = X(x)T(t) \), and formulate the ordinary differential equations and boundary conditions for \( X \) and \( T \). (b) Solve the eigenvalue problem and find the partial solutions, \( u_n(x,t) \). (c) Which of the equations (1-3) do these solutions satisfy? (d) Write down the solution of the full problem as an infinite series.

**Solution.**

(a) We present it as \( u(x,t) = X(x)T(t) \) and obtain

\[
T'X - kTX'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{kT} = -\lambda.
\]

This gives rise to two ordinary differential equations. For \( X \) we have the following boundary value problem,

\[
X'' + \lambda X = 0, \quad X(0) = X(\pi) = 0,
\]

and for \( T \) we have

\[
T' + 2\lambda T = 0.
\]

(b) The problem for \( X \) is an eigenvalue problem, it only has nontrivial solutions if \( \lambda = n^2 \) for integer values of \( n \), and the solutions are

\[
X_n(x) = \sin nx.
\]

The problem for \( T \) becomes \( T' + 2n^2 T = 0 \), and its general solution is \( T(t) = Ce^{-2n^2 t} \). The partial solutions of the whole problem for \( u(x,t) \) are

\[
u_n(x,t) = \sin nx e^{-2n^2 t}.
\]

(c) These partial solutions satisfy equations (1) and (2), but not (3).

(d) To satisfy the remaining boundary condition, equation (3), we present the solution as an infinite sum,

\[
u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx e^{-2n^2 t}.
\]

We have

\[
u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{\sin nx}{n!},
\]
therefore we choose \[ b_n = \frac{1}{n!}, \]
and we have
\[ u(x, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \sin nx e^{-2n^2t}. \]

6. Consider the following boundary value problem,
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad (4) \]
\[ u(x, 0) = u(0, y) = 0, \quad (5) \]
\[ u(x, \pi) = \sin 5x, \quad (6) \]
\[ u(\pi, y) = \sin 7y. \quad (7) \]

(a) Use linear decomposition \( u(x, t) = u^{(1)}(x, y) + u^{(2)}(x, y) \) and formulate the problems for \( u^{(1)} \) and \( u^{(2)} \) which can be solved by separation of variables. 
(b) Solve the problems for \( u^{(1)} \) and \( u^{(2)} \) by separation of variables.  
(c) Write down the solution \( u(x, t) \).

**Solution.** (a) Let us set \( u(x, t) = u^{(1)}(x, y) + u^{(2)}(x, y) \) such that the function \( u^{(1)}(x, y) \) satisfies
\[ \frac{\partial^2 u^{(1)}}{\partial x^2} + \frac{\partial^2 u^{(1)}}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad (8) \]
\[ u^{(1)}(x, 0) = u^{(1)}(0, y) = u^{(1)}(x, \pi) = 0, \quad (9) \]
\[ u^{(1)}(x, \pi) = \sin 5x, \quad (10) \]
and the function \( u^{(2)}(x, y) \) satisfies
\[ \frac{\partial^2 u^{(2)}}{\partial x^2} + \frac{\partial^2 u^{(2)}}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad (11) \]
\[ u^{(2)}(x, 0) = u^{(2)}(0, y) = u(x, \pi) = 0, \quad (12) \]
\[ u^{(2)}(\pi, y) = \sin 7y. \quad (13) \]

Then, by adding \( u = u^{(1)} + u^{(2)} \) we can see that the function \( u(x, y) \) satisfies the original problem.
(b) Let us solve the problem for \( u^{(1)}(x, y) \), equations (8-10). We present it as \( u^{(1)}(x, y) = X(x)Y(y) \) and obtain

\[
X''Y + Y''X = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.
\]

This gives rise to two ordinary differential equations, each with its boundary conditions. For \( X \) we have

\[
X'' + \lambda X = 0, \quad X(0) = X(\pi) = 0,
\]

and for \( Y \) we have

\[
Y'' - \lambda Y = 0, \quad Y(0) = 0.
\]

The problem for \( X \) is an eigenvalue problem, it only has nontrivial solutions if \( \lambda = n^2 \) for integer values of \( n \), and the solutions are

\[
X_n(x) = \sin nx.
\]

The problem for \( Y \) becomes \( Y'' - n^2 Y = 0 \), and its general solution is \( Y(y) = Ce^{ny} + De^{-ny} \). Using the boundary condition at \( y = 0 \) we obtain,

\[
Y_n(y) = \sinh ny.
\]

The partial solutions of the whole problem for \( u^{(1)} \) are

\[
u_n^{(1)} = \sin nx \sinh ny.
\]

To satisfy the remaining boundary condition, equation (10), we present the solution as an infinite sum,

\[
u^{(1)}(x, y) = \sum_{n=1}^{\infty} c_n \sin nx \sinh ny.
\]

From

\[
u^{(1)}(x, \pi) = \sin 5x
\]

we obtain

\[
\sum_{n=1}^{\infty} c_n \sin nx \sinh n\pi = \sin 5x.
\]
Let us define $b_n = c_n \sinh n\pi$. Then we have

$$\sum_{n=1}^{\infty} b_n \sin nx = \sin 5x \quad \Rightarrow \quad b_n = 0 \quad \forall n \neq 5, \quad b_5 = 1.$$ 

Therefore, $c_n = 0 \quad \forall n \neq 5, \quad c_5 = 1/\sinh 5\pi$, and we obtain the solution

$$u^{(1)}(x, y) = \frac{\sin 5x \sinh 5y}{\sinh 5\pi}.$$

Next, let us solve the problem for $u^{(2)}(x, y)$, equations (11-13). We present it as $u^{(1)}(x, y) = X(x)Y(y)$ and obtain

$$X''Y + Y''X = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$ 

This gives rise to two ordinary differential equations, each with its boundary conditions. For $X$ we have

$$X'' + \lambda X = 0, \quad X(0) = 0,$$

and for $Y$ we have

$$Y'' - \lambda Y = 0, \quad Y(0) = 0, \quad Y(\pi) = 0.$$ 

The problem for $Y$ is an eigenvalue problem, so we solve that first. It only has nontrivial solutions if $\lambda = -n^2$ for integer values of $n$, and the solutions are

$$Y_n(y) = \sin ny.$$ 

The problem for $X$ becomes $X'' - n^2 X = 0$, and its solution is found as before, only this time it’s $X$ and not $Y$:

$$X_n(y) = \sinh nx.$$ 

The partial solutions of the whole problem for $u^{(2)}$ are

$$u^{(2)}_n = \sin ny \sinh nx.$$ 

To satisfy the remaining boundary condition, equation (13), we present the solution as an infinite sum,

$$u^{(2)}(x, y) = \sum_{n=1}^{\infty} c_n \sin ny \sinh nx.$$
From 
\[ u^{(2)}(\pi, y) = \sin 7y \]
we obtain
\[ \sum_{n=1}^{\infty} c_n \sin ny \sinh n\pi = \sin 7y. \]

Let us define \( b_n = c_n \sinh n\pi \). Then we have
\[ \sum_{n=1}^{\infty} b_n \sin ny = \sin 7x \Rightarrow b_n = 0 \quad \forall n \neq 7, \quad b_7 = 1. \]

Therefore, \( c_n = 0 \quad \forall n \neq 7, \quad c_7 = 1/\sinh 7\pi \), and we obtain the solution
\[ u^{(2)}(x, y) = \frac{\sin 7y \sinh 7x}{\sinh 7\pi}. \]

(c) The solution of the full original problem is given by
\[ u(x, t) = \frac{\sin 5x \sinh 5y}{\sinh 5\pi} + \frac{\sin 7y \sinh 7x}{\sinh 7\pi}. \]