Abstract

Class notes on vectors, linear combination, basis, span.

1 Vectors

Vectors on the plane are ordered pairs of real numbers \((a, b)\) such as \((0, 1), (1, 0), (1, 2), (-1, 1)\). The plane is denoted by \(\mathbb{R}^2\), also known as Euclidean 2-space.

Vectors in our physical space are ordered triples \((a, b, c)\) such as \((1, 0, 0), (0, 0, 1), (0, 1, 0), (1, 1, 2), (1, -1, 2)\). All such vectors form Euclidean 3-space, or \(\mathbb{R}^3\).

In general, the set of all ordered \(n\)-tuples \((x_1, x_2, \ldots, x_n)\) is Euclidean \(n\)-space.

1.1 Addition and Scalar Multiplication

Let \(v = [v_1, v_2, \ldots, v_n]\) and \(w = [w_1, w_2, \ldots, w_n]\) be two vectors in \(\mathbb{R}^n\). Then vector addition is:

\[ v + w = [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n], \]

vector subtraction is:

\[ v - w = [v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n]. \]

For any number \(r\) (scalar), vector scalar multiplication is:

\[ r v = [r v_1, r v_2, \ldots, r v_n]. \]

Two nonzero vectors \(v\) and \(w\) in \(\mathbb{R}^n\) are said to be parallel if one is a scalar multiplication of the other, \(v = r w\). If \(r > 0\) (\(r < 0\)), they point in the same (opposite) direction.

Example 1: are the two vectors \(v = [2, 1, 3, -5]\), and \(w = [6, 3, 9, -15]\) parallel? Yes, \(w = 3v\).
1.2 Linear Combination

Given vectors \( v^1, v^2, \ldots, v^k \) in \( \mathbb{R}^n \), and scalars \( r_1, r_2, \ldots, r_k \) in \( R \), the vector:

\[
 r_1 v^1 + r_2 v^2 + \cdots + r_k v^k, \tag{1.1}
\]

is called a linear combination of the vectors \( v^1, v^2, \ldots, v^k \) with scalar coefficients \( r_1, r_2, \ldots, r_k \).

**Example 2**: any vector \([a_1, a_2]\) in \( \mathbb{R}^2 \) can be expressed as a unique linear combination of the two vectors \([1, 0]\) and \([0, 1]\):

\[
 [a_1, a_2] = r_1[1, 0] + r_2[0, 1], \tag{1.2}
\]

if and only if \( r_1 = a_1, r_2 = a_2 \). We call \([1, 0]\) and \([0, 1]\) standard basis vectors of \( \mathbb{R}^2 \), often denoted by \( e^1 \) and \( e^2 \). Similarly, standard basis vectors in \( \mathbb{R}^3 \) are:

\[
 e^1 = [1, 0, 0], \quad e^2 = [0, 1, 0], \quad e^3 = [0, 0, 1].
\]

Standard basis vectors of \( \mathbb{R}^n \) are \( e^1, e^2, \ldots, e^n \), where \( e^j, 1 \leq j \leq n \), is the vector with zero components except that the \( j \)-th component equals 1. Each vector in \( \mathbb{R}^n \) is a unique linear combination of the standard basis vectors.

1.3 Span

Let \( v^1, v^2, \ldots, v^k \) be vectors in \( \mathbb{R}^n \). The span of these vectors is the set of all linear combinations of them, and is denoted by \( \text{sp}(v^1, v^2, \ldots, v^k) \).

**Example 3**: the span of \([1, 0]\) and \([0, 1]\) is \( \mathbb{R}^2 \). The span of \( v^1 = [1, -2] \) and \( v^2 = [7, -14] \) is the line along \([1, -2]\) instead of \( \mathbb{R}^2 \), because \( v^2 \) is a scalar multiple of \( v^1 \) (or \( v^2 \) is in \( \text{sp}(v^1) \)).

**Example 4**: Let \( v = [1, 3], w = [-2, 5] \), determine if \([-1, 19]\) is in \( \text{sp}(v, w) \) and if so the coefficients of linear combination.

The problem is same as finding a solution to \([1, -19] = r v + s w = r[1, 3] + s[-2, 5] \) for two real numbers \( r \) and \( s \). It follows from comparing the components that:

\[
 r - 2s = -1, \quad 3r + 5s = 19.
\]

Eliminating the \( r \) variable gives:

\[
 0 + 11s = 22, \quad s = 2, \quad r = 3. \quad \text{Solution is unique.}
\]
1.4 Matlab Exercises

Here are hands-on Matlab Exercises.

Exercise 1: enter vectors in Matlab:

\[ v = [1 \ 2 \ 3 \ 4] \]

\[ w = [1 \ 1 \ 1 \ 1] \]

\[ v + w \]

\[ v - w \]

\[ 2 \times v \]

\[ -3 \times v \]

\[ 2 \times v + w \]

\[ \text{zeros}(1,10) \]

\[ \text{ones}(1,10) \]

\[ u = [v \ w]; \text{plot}(u, ’r*’) \]

\[ \text{axis}([0 \ 10 \ -1 \ 5]) \]

add ”title(’plotting an 8-dimensional vector’”).

References