Introduction to Linear Algebra III

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Abstract

Linear system, matrix and matrix operations, row echelon form, rank.

1 Linear System and Matrix

A linear system:

\[ \begin{aligned}
  x_1 + 2x_2 &= 1, \\
  3x_1 + 5x_2 &= 9,
\end{aligned} \]  

(1)

can be put in vector form:

\[ x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \]  

(2)

which is abbreviated into matrix form:

\[ \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \]  

(3)

Matrix is a collection of \( m \) row (\( n \) column) vectors of dimension \( n \) (\( m \)), or an array of \( m \times n \) entries of real or complex numbers. It was first found in J. Sylvester’s work in 1850, and it meant a place where something is bred, produced and developed.

Product of matrix \( A = [a_{ik}]_{m \times n} \) and matrix \( B = [b_{kj}]_{n \times s} \) is a matrix \( AB = C = [c_{ij}]_{m \times s} \) where the entry \( c_{ij} \) is the dot product of the \( i \)-th row of \( A \) with \( j \)-th column of \( B \), or:

\[ c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \]

In Matlab, enter:

\[ A = [0 2; 3 5], \ B = [0 1; 2 5], \ A * B, \ B * A, \]

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and compare $A \times B$ with $B \times A$. In general, matrix products do not commute.

Matrix addition (difference) is done entry by entry for matrices of same dimensions. Matlab notations are $A \pm B$. Scalar multiple is done entrywise too, e.g. $3 \times A$, $4 \times B$. A matrix linear combination is: $2 \times A - 5 \times B$.

Matrix transpose $A^T$ is a matrix obtained from $A$ by the transform $a_{ij} \rightarrow a_{ji}$. In Matlab, $A'$ will do. If $A$ contains complex entries, then prime also includes conjugate: $a_{ij} \rightarrow \text{conj}(a_{ji})$. Enter in Matlab: $D = A + i \times B$, then $D'$ and compare with $D$.

A matrix is called symmetric matrix if $A = A^T$. A few properties of matrices are:

- Commutative law, $A + B = B + A$;
- Associative law: $(A + B) + C = A + (B + C)$, $(AB)C = A(BC)$;
- Distributive law: $A(B + C) = AB + AC$, $(A + B)C = AC + BC$;
- Scalar pulls through: $(sA)B = A(sB) = sAB$;

Example 1: in Matlab, mesh(A) plots entry values of matrix $A$ as a function of row and column numbers. To plot a function of two variables, $z = 9 - x^2 - y^2$, $x \in [-3, 3]$, $y \in [-3, 3]$, define:

$$[x, y] = \text{meshgrid}(-3:0.5:3, -3:0.5:3),$$

$$z = 9 - x^2 - y^2,$$

then mesh(z). You may also plot with “surf(z)”.

2 Square Matrices and Inverse

A square $n \times n$ matrix $A$ is invertible if there exists a square $n \times n$ matrix $C$ such that $CA = AC = I_n$, $I_n$ the $n \times n$ identity matrix. The matrix $C$ is the inverse of $A$, denoted by $A^{-1}$. Otherwise, $A$ is called singular.

The inverse is always unique if it exists. Suppose $AC = DA = I_n$, then $C = D$. In fact, by the associative law and the assumption, we have:

$$D = D(AC) = (DA)C = C.$$
Next, we show that if \( AC = CA = I_n \), then \( C \) is unique. Suppose the contrary, \( AD = DA = I_n \) for some matrix \( D \), then by (4), \( D = C \).

**Theorem 2.1.** \((AB)^{-1}\) exists if \( A \) and \( B \) are invertible, moreover \((AB)^{-1} = B^{-1}A^{-1}\).

One can verify the inversion of \( AB \) by direct multiplication. The right inverse is from the identity:
\[
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I_n.
\]
The left inverse is similar.

In Matlab, inverse is \( A^{-1} \), or \( \text{eye}(n)/A \).

### 3 Row Reduction

Finding inverse is related to solving a linear system by row reduction (or Gauss elimination of variables). Elementary row operations consist of:

- two row exchange: \( R_i \rightarrow R_j \), where \( R_i \) is the \( i \)-th row vector;
- row scaling: \( R_i \rightarrow s R_i \);
- row addition: \( R_i \rightarrow R_s + s R_j \).

We shall apply a sequence of elementary row reductions to augmented matrix \([A \ b]\) to obtain a simpler (sparse) matrix \([H \ c]\) with many entries equal to zero. Since row operations do not alter the solution, \( Hx = c \) has the same solution as \( Ax = b \). How simple is \( H \) ? The simple form is called reduced row echelon form (RREF).

RREF can be described as follows:

- all zero rows appear below rows with nonzero entries;
- first nonzero entry in any row (called pivot) appears in a column to the right of the first nonzero entry (pivot) in any proceeding row;
- all pivots equal to 1 with zeros above and below each pivot.

If the last condition is not required, the resulting simplified matrix is called row echelon form. Each elementary row operation is same as multiplying from the left by an elementary
matrix. For example, exchanging second and third rows in a matrix with 3 rows can be achieved by left-multiplication with matrix:

\[ E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \].

The matrix for the first row scaling is:

\[ E_1 = \begin{bmatrix} 1 & 0 & 0 \\ s & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

The matrix for adding 2 times the first row to the second is:

\[ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

RREF comes from the left product of a sequence of elementary matrices onto a matrix \( A \). Elementary matrices are invertible because they can be turned into identity by reversing its associated row operations. The inverse of an elementary matrix is still an elementary matrix. If \( A \) is a square matrix, it is invertible if and only if its RREF is identity. The product of the sequence of elementary matrices is just the inverse.

To summarize, the following conditions are equivalent:

- an \( n \times n \) matrix \( A \) is invertible;
- the RREF of \( A \) is identity;
- column vectors of \( A \) span \( \mathbb{R}^n \);
- \( A \) is a product of elementary matrices;
- \( Ax = b \) has unique solution for any vector \( b \in \mathbb{R}^n \).

Matlab command \([R, j] = \text{rref}(A)\) gives the RREF in \( R \) and pivot column numbers in \( j \).

Example 2: find inverse of the matrix in Matlab:

\[ A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 3 & 4 & -7 \end{bmatrix} \]
by $A^{-1}$, or by $\text{rref}([A \ \text{eye}(3)])$ which is in the block form $[\text{eye}(3) \ A^{-1}]$.

**Example 3:** compute RREF of the augmented matrix:

$$A = \begin{bmatrix}
1 & 2 & 1 & | & 1 \\
2 & -1 & 1 & | & 2 \\
4 & 3 & 3 & | & 4 \\
3 & 1 & 2 & | & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 3/5 & | & 1 \\
0 & 1 & 1/5 & | & 0 \\
0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & | & 0
\end{bmatrix}$$

The corresponding linear system is over-determined, because there are 4 equations and 3 unknowns. However, it is consistent thanks to the two zero rows in RREF. The number of pivots is equal to two, which is called the rank of the matrix, $\text{rank}(A)=2$. The right hand side vector is in the span of the column vectors of the coefficient matrix (submatrix in $A$, on the left of the dashed vertical line). There are infinitely many solutions. For any $x_3$,

$$x_1 = 1 - 3x_3/5, \ x_2 = -x_3/5.$$  

**Example 4:** consider an under-determined system:

$$x_1 + 2x_2 + x_3 = 1,$$

$$2x_1 + 4x_2 + 2x_3 = 3,$$

whose augmented matrix is:

$$B = \begin{bmatrix}
1 & 2 & 1 & | & 1 \\
2 & 4 & 2 & | & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 1 & | & 1 \\
0 & 0 & 0 & | & 1
\end{bmatrix}$$

which is inconsistent, and has no solution.

Consistent under-determined system has infinitely many solutions:

**Example 5:** consider the augmented matrix:

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & | & 2 \\
1 & 1 & 1 & 2 & 2 & | & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & | & 1 \\
0 & 0 & 0 & 1 & 0 & | & 2 \\
0 & 0 & 0 & 0 & 1 & | & -1
\end{bmatrix}$$

For any $(x_2, x_3)$,

$$x_1 = 1 - x_2 - x_3, \ x_4 = 2, \ x_5 = -1.$$  

The coefficient matrix (on the left of the dashed vertical line in $A$) has rank =3, there are two free parameters in the solution vector in $\mathbb{R}^5$. Rank plus the number of free parameters equal the number of columns of the coefficient matrix.

**References**