Math 112B final problems

1. Solve the initial value problem $\frac{\partial u}{\partial t} + \cosh t \frac{\partial u}{\partial x} = xu + t$, $u(x, 0) = \sin x^2$.

**Solution.** Let us suppose we want to dfind the solution $u(x_0, t_0)$ at a particular point, $(x_0, t_0)$. We solve the following system for $x$ and $t$:

\[
\begin{align*}
\frac{dt}{dr} &= 1, \\
\frac{dx}{dr} &= \cosh t.
\end{align*}
\]

From the first equation we obtain $t = r + c$. We use the condition $r = 0, t = 0$ to obtain $t = r$. The second equation gives us $x = \sinh r + \xi = \sinh t + \xi$. To find the constant $\xi$, we use the condition that the characteristic passes through point $(x_0, t_0)$: $x_0 = \sinh t_0 + \xi$, which gives us

\[\xi = x_0 - \sinh t_0.\]

The equation for the characteristic is

\[x(t) = \sinh t + x_0 - \sinh t_0.\]

We will need the point, $x = x_\ast$, where the characteristic line crosses the $x$ axis. This is given by

\[x(0) = x_0 - \sinh t_0 = x_\ast.\]

Next, we consider the equation for $u$:

\[\frac{du}{dr} = xu + t.\]

Plug $t = r$ and $x = \sinh r + \xi$ into this equation to get

\[\frac{du}{dr} = (\sinh r + \xi)u + r.\]

This has to be equipped with the boundary condition for $u$:

\[u(0) = \sin x_\ast^2.\]
To solve the equation above, we first consider the homogeneous case \( \frac{du}{dr} = (\sinh r + \xi)u \), i.e., \( \frac{du}{dr} = A e^{\cosh r + \xi} \). To solve the non-homogeneous case, we let \( u(r) = A(r) e^{\cosh r + \xi} \), then \( \frac{du}{dr} = \frac{dA}{dr} e^{\cosh r + \xi} + A(r) e^{\cosh r + \xi} (\sinh r + \xi) \). Therefore, we have
\[
\frac{dA}{dr} = re^{-\cosh r - \xi}.
\]
Solving this equation for \( A \), we get \( A(r) = \int re^{-\cosh r - \xi} dr + A_0 \). Then we substitute \( A(r) \) into \( u(r) = A(r) e^{\cosh r + \xi} \) to have
\[
(\int r' e^{-\cosh r' - \xi} dr' + A_0)e^{\cosh r + \xi}.
\]
Finally, we need to find the constant \( A_0 \) from the boundary condition for \( u \), equation (1):
\[
u(0) = A_0 e^{1+0} = A_0 e = \sin x^2,
\]
which gives
\[
A_0 = \frac{\sin x^2}{e}.
\]
Using this in equation (2), we find \( u(x_0, t_0) \) (by replacing \( r \rightarrow t_0 \)):
\[
u(x_0, t_0) = \left(\int_0^{t_0} r e^{-\cosh r' - (x_0 - \sinh t_0)} e^{r'} dr' + \frac{\sin (x_0 - \sinh t_0)^2}{e}\right)e^{\cosh t_0 + (x_0 - \sinh t_0)t_0}.
\]

2. Solve the initial value problem \( \frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0 \), \( u(x, 2) = \frac{1}{1+x^2} \).

**Solution.** We solve the following system for \( x \) and \( t \)
\[
\frac{dt}{dr} = t,
\frac{dx}{dr} = 2.
\]
From the first equation we obtain \( \ln t = r + c \). We use the condition \( r = 0, t = 2 \) to get \( \ln \frac{1}{2} = r \). The second equation gives us \( x = 2r + \xi \), then we have \( x - 2\ln \frac{1}{2} = \xi \). The general solution is \( u(x, t) = p(x - 2\ln \frac{1}{2}) \). Therefore, \( u(x, 2) = p(x) = \frac{1}{1+x^2} \), which implies \( u(x, t) = \frac{1}{1+(x-2\ln \frac{1}{2})^2} \).
3. Consider the inviscid Burgers’ equation with the initial condition \( u(x, 0) = u_0(x) \). (a) Find the solution \( u(x, t) \) by using the explicit formula, if \( u_0(x) = x/2+1 \). (b) Find equations for the characteristics and draw the characteristics in the \((x,t)\) plane, for \( u_0 = 1 \) with \( x < 0 \), \( u_0 = 1 - x \), \( 0 \leq x \leq 1 \), and \( u_0 = 0 \) with \( x > 1 \). (c) Will a shock form in the case with \( u_0(x) \) as in (b)? If so, find the earliest time of shock formation.

Solution:

(a) We have \( u = u_0(x - ut) \), to be resolved for the unknown \( u \). Using \( u_0(x) = x + 1 \) we have \( u = (x - ut)/2 + 1 \), which gives \( u(x, t) = \frac{x+2}{t+2} \). (b) The characteristics have equations \( x = t + c \) for \( c < 0 \), \( x = (1 - c)t + c \) with \( 0 \leq c \leq 1 \), and \( x = c \) (horizontal lines) with \( c > 1 \). Note that all the characteristics with \( 0 \leq c \leq 1 \) cross at the point \((1,1)\). (c) Since the initial condition \( u_0(x) \) has a negative slope, a shock will form. The earliest time of shock formation is \( t^* = \min_x \left( -\frac{1}{u'_0(x)} \right) \), for all \( x \) where \( u'(x) < 0 \). We have \( u'_0 = -1 \), therefore \( t^* = 1 \).

4. Consider the inviscid Burgers’ equation with the initial condition \( u(x, 0) = u_0(x) \). (a) Will a shock form if \( u_0(x) = 1 + \tanh(x) \)? (b) Same with \( u_0(x) = 1 - \tanh(x) \)? Explain. If possible, find the earliest time of shock formation.

Solution:

(a) No shock will form because \( du_0(x)/dx = 1/\cosh^2(x) > 0 \) for all \( x \). (b) A shock will form in finite time because \( du_0(x)/dx = -1/\cosh^2(x) < 0 \) for all \( x \). This happens at time

\[
t^* = \min_x \left( -\frac{1}{u'_0(x)} \right) = \min_x (\cosh^2(x)) = 1.
\]

5. Consider the nonhomogeneous ODE:

\[
\begin{align*}
    u'' - 25u &= -f(x), & 0 < x < 2, \\
    u'(0) &= 2, & u(2) = 0.
\end{align*}
\]
(a) Solve the ODE. (b) Plot the Green’s function \( G(x, \xi) \) as a function of \( x \) for a fixed point \( \xi \). What does it represent?

**Solution.** (a) Solve for Green’s function \( G(x, \xi) \). For fixed \( \xi \), \( G \) satisfies

\[
\begin{cases}
\frac{d^2 G}{dx^2} - 25G = 0 & \text{for } x \neq \xi, \\
\frac{dG}{dx} \bigg|_{x=0} = G \bigg|_{x=2} = 0 \\
G \bigg|_{x=\xi+0} - G \bigg|_{x=\xi-0} = 0 \\
\frac{dG}{dx} \bigg|_{x=\xi+0} - \frac{dG}{dx} \bigg|_{x=\xi-0} = \frac{-1}{p(\xi)} = -1
\end{cases}
\]

From the first two conditions we have that

\[G(x, \xi) = \begin{cases} 
  c(\xi) \cosh (5x), & x < \xi, \\
  d(\xi) \sinh (5(x-2)), & x > \xi,
\end{cases}\]

where \( c \) and \( d \) depend on \( \xi \) (but are independent of \( x \)). The symmetry condition on \( G \) (that is, \( G(x, \xi) = G(\xi, x) \)) implies that

\[G(x, \xi) = \begin{cases} 
  A \sinh (5(\xi - 2)) \cosh (5x), & x < \xi, \\
  A \cosh (5\xi) \sinh (5(x-2)), & x > \xi,
\end{cases}\]

where \( A \) is independent of both \( x \) and \( \xi \). Now the final condition in (5) tells us that

\[5A \cosh (5\xi) \cosh (5(\xi - 2)) - 5A \sinh (5(\xi - 2)) \sinh (5\xi) = -1\]

\[\Rightarrow \quad A = \frac{-1}{5 \cosh 10}.
\]

Thus

\[G(x, \xi) = \begin{cases} 
  \frac{-1}{5 \cosh 10} \sinh (5(\xi - 2)) \cosh (5x), & x < \xi, \\
  \frac{-1}{5 \cosh 10} \cosh (5\xi) \sinh (5(x-2)), & x > \xi.
\end{cases}\]

Now our solution is

\[u(x) = \int_0^2 f(\xi)G(x, \xi)d\xi + c_1e^{5x} + c_2e^{-5x}.
\]
The last two terms, two linearly independent solutions to the homogeneous equation, are necessary since we have non-zero boundary conditions. Solve for $c_1$ and $c_2$ by plugging in the boundary conditions (the integral part will be zero when you plug in). Doing so results in $c_1 = (5e^{10} \cosh 10)^{-1}$ and $c_2 = (5e^{10} \cosh 10)^{-1} - \frac{2}{5}$. Then plugging our expression for $G$ into the integral, we have our final answer

$$u(x) = \int_0^x f(\xi) \frac{-1}{5 \cosh 10} \cosh (5\xi) \sinh (5(x - 2)) d\xi$$

$$+ \int_x^2 f(\xi) \frac{-1}{5 \cosh 10} \sinh (5(\xi - 2)) \cosh (5x) d\xi$$

$$+ (5e^{10} \cosh 10)^{-1} e^{5x} + ((5e^{10} \cosh 10)^{-1} - \frac{2}{5}) e^{-5x}.$$

6. Consider the Poisson equation in a square,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin 2x \sin 3y, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad (6)$$

$$u(x, y) = 0 \quad \text{on the boundary} \quad (7)$$

(a) Reduce the problem to inhomogeneous ODEs. (b) Solve the ODEs. (c) Present the solution $u(x, y)$ is a series form.

Solution. (a) Recall that in the homogeneous case of this equation, separation of variables gives us the eigenvalue problem

$$X'' + \lambda X = 0, X(0) = X(\pi) = 0$$

which has solutions $X_n = \sin nx$. This motivates the idea to look for a solution $u$ in the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n(y) \sin nx \quad (8)$$

where $b_n(y) = \frac{2}{\pi} \int_0^\pi u(x, y) \sin nx dx$. We now take the sine transform of the whole equation.

$$\frac{2}{\pi} \int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin nx dx + \frac{2}{\pi} \int_0^\pi \frac{\partial^2 u}{\partial y^2} \sin nx dx = \frac{2}{\pi} \int_0^\pi -\sin 2x \sin 3y \sin nx dx$$

$$\Rightarrow -n^2 b_n(y) + b_n''(y) = -F_n(y).$$

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The first term comes from integrating by parts, and the second term from switching the derivative with the integral (which we can do because the derivative is on the $y$ variable, but the integration is in $x$). We can also calculate the right hand side.

\[
F_n(y) = \frac{2}{\pi} \int_0^\pi \sin 2x \sin 3y \sin nxdx
\]

\[
= \sin 3y \frac{2}{\pi} \int_0^\pi \sin 2x \sin nxdx
\]

\[
= \begin{cases} 
\sin 3y, & \text{if } n = 2, \\
0, & \text{if } n \neq 2.
\end{cases}
\]

(The easy way to do that integral is to recognize that \(\frac{2}{\pi} \int_0^\pi \sin 2x \sin nxdx\) are the sine series coefficients of the function \(\sin 2x\).)

As for boundary conditions, \(b_n(0) = \frac{2}{\pi} \int_0^\pi u(x, 0) \sin nxdx = 0\), and similarly \(b_n(\pi) = 0\). Thus our problem reduces to the following system of ODEs:

\[
\begin{cases}
\begin{align*}
& b''_2(y) - 4b_2(y) = -\sin 3y, \quad 0 < x < \pi, \quad 0 < y < \pi, \\
& b_2(0) = b_2(\pi) = 0; \\
& b''_n(y) - n^2b_n(y) = 0, \\
& b_n(0) = b_n(\pi) = 0, \quad n = 1, 3, 4, \ldots
\end{align*}
\end{cases}
\]

(b) When \(n \neq 2\), clearly \(b_n(y) \equiv 0\). To solve the equation for \(n = 2\), we find

the Green’s function in the same manner as in problem 1. We get

\[
G(y, \eta) = \begin{cases} 
\frac{1}{2 \sinh 2\pi} \sinh 2(\pi - \eta) \sinh 2y, & \text{for } y < \eta, \\
\frac{1}{2 \sinh 2\pi} \sinh 2\eta \sinh 2(\pi - y), & \text{for } y > \eta.
\end{cases}
\]

Then

\[
b_2(y) = \int_0^\pi G(y, \eta) \sin 3\eta d\eta = \frac{1}{2 \sinh 2\pi} \left[ \sinh 2(\pi - y) \int_0^y \sinh 2\eta \sin 3\eta d\eta \\
+ \sinh 2y \int_y^\pi \sinh 2(\pi - y) \sin 3\eta d\eta \right].
\]
(c) Plugging the solutions of our ODEs into (8), we have

\[ u(x, y) = b_2(y) \sin 2x \]

\[ = \frac{1}{2 \sinh 2\pi} \sin 2x \left[ \sinh 2(\pi - y) \int_0^y \sinh 2\eta \sin 3\eta d\eta + \sinh 2y \int_y^\pi \sinh 2(\pi - \eta) \sin 3\eta d\eta \right]. \]