Math 112A practice problems with solutions

ATTENTION: For the Midterm you will be required to show your calculations and explain the reasoning. An answer alone is not enough. Also, you will have to sketch graphs (which I didn’t do here for technical reasons).

1. Consider the wave equation on an infinite line, with $u(x,0) = f(x)$ defined by

$$f(x) = \begin{cases} 
0, & x < 0, \\
x^2, & 0 \leq x \leq 1, \\
(2-x), & 1 \leq x \leq 2, \\
0, & x > 2.
\end{cases}$$

Set $\partial u/\partial t(x,0) = g(x) = 0$ and $c = 1/2$. Draw the solution at $t = 0$ and $t = 5$. Calculate the time, $t$, at which $u(15,t) = 1/2$.

Solution: $u(x,0)$ is simply given by $f(x)$. $u(x,5)$ is given by $1/2[f(x + 5/2) + f(x - 5/2)]$. This looks like $2$ humps of height $1/2$, each of them has the shape of the initial condition and the maxima are located at points $x = -3/2$ and $x = 7/2$. Finally, $u(15,t) = 1/2$ for $t = 28$.

2. Suppose that the string of problem 1 is finite, with boundaries at $x = 0$ and $x = 5$:

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{4} \frac{\partial^2 u}{\partial x^2} = 0,$$

$u(x,0) = f(x)$, see Problem 1,

$$\frac{\partial u}{\partial t}(x,0) = 0,$$

$u(0,t) = 0$,

$u(5,t) = 0$.

The initial condition has a cusp at $x = 1$. On an $x$-$t$ diagram (with $0 \leq t \leq 4$) show how this cusp will propagate. (Hint: cusps propagate along characteristics.)

Solution: Draw the $x$-$t$ diagram, with $2$ characteristics. One of them has the equation $t = 2 - 2x$ for $0 \leq x \leq 1$ and then it is reflected from the left wall with the equation $t = 2 + 2x$, $0 \leq x \leq 1$. The other one is $t = -2 + 2x$.
for $1 \leq x \leq 3$. The intersection of these characteristics with horizontal lines indicate the position of the cusp on the string at corresponding moments of time.

3. A string of length 3 is fixed at the right end, while the left end is a "sliding loop" (free). No external force is applied. Set $c=2$ and write down the initial-boundary value problem with the initial shape given by $x^2(3-x)$ and the initial velocity $x^2$. Draw the extension of the initial data outside the string. Also, find $u(1,4)$ and $u(3,10)$.

Solution: The equations are

$$\frac{\partial^2 u}{\partial t^2} - 4 \frac{\partial^2 u}{\partial x^2} = 0,$$

$$u(x,0) = x^2(3-x),$$

$$\frac{\partial u}{\partial t}(x,0) = x^2,$$

$$\frac{\partial u}{\partial x}(0,t) = 0,$$

$$u(3,t) = 0,$$

The extensions of both initial shape and initial velocity are drawn by an even reflection around $x = 0$ and an odd reflection around $x = 3$.

Define $f(x) = x^2(3-x)$ for $0 \leq x \leq 3$ and $g(x) = x^2$ for $0 \leq x \leq 3$. Since we have a free boundary condition at $x = 0$, we need to extend $f$ and $g$ so that they are even functions (i.e., $f(x) = f(-x)$ and $g(x) = g(-x)$) and so that they are odd functions about $\ell$ (i.e., $f(x) = -f(2\ell - x)$ and $g(x) = -g(2\ell - x)$). Then we can substitute these extended functions into the d’Alembert solution of the wave equation.

To find $u(1,4)$, notice $x + ct = 9$ and $x - ct = -7$. Using what we know about $f$, we have $f(9) = -f(2\ell - 9) = -f(-3) = -f(3) = 0$ and $f(-7) = f(7) = -f(2\ell - 7) = -f(-1) = -f(1) = -2$. Similarly for $g$, we have $\int_{-7}^{0} g(x) \, dx = -\int_{-1}^{3} g(x) \, dx = -\int_{-1}^{3} x^2 \, dx = -\frac{28}{3}$. Therefore, $u(1,4) = \frac{1}{2} (0 - 2) + \frac{1}{2c} \left( -\frac{28}{3} \right) = -10/3$.

Using the fixed boundary condition from above, we have $u(3,10) = 0$. 
4. Consider a wave equation on an infinite line,
\[
\frac{\partial^2 u}{\partial t^2} - 9 \frac{\partial^2 u}{\partial x^2} = 0.
\]
Find the characteristics though the point \((1, 3)\). Draw the domains of dependence and influence of the point \((1, 3)\).

**Solution:** The equations of characteristics are \(t = (10 - x)/3\) and \(t = (x + 8)/3\). The domain of dependence is below the two intersecting lines. The domain of influence is above.

5. Given the initial boundary value problem, for \(0 \leq x \leq 1\),
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - 4 \frac{\partial^2 u}{\partial x^2} &= 0, \\
u(x, 0) &= x(1 - x), \\
\frac{\partial u}{\partial t}(x, 0) &= \sqrt{x}, \\
u(0, t) &= t, \\
u(1, t) &= \sin t,
\end{align*}
\]
write down the solution as \(u(x, t) = v(x, t) + w(x, t)\), where \(v(x, t)\) satisfies the boundary conditions and \(w(x, t)\) can be found as the D’Alembert solution of a problem with homogeneous boundary conditions. Find \(v(x, t)\) explicitly. Formulate the problem for \(w(x, t)\) (DO NOT SOLVE FOR \(w(x, t)\)).

**Solution:** \(v(x, t) = x \sin t + (1 - x)t\). The problem for \(w\) is:
\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2} - 4 \frac{\partial^2 w}{\partial x^2} &= x \sin t, \\
w(x, 0) &= x(1 - x), \\
\frac{\partial w}{\partial t}(x, 0) &= \sqrt{x} - 1, \\
w(0, t) &= 0, \\
w(1, t) &= 0.
\end{align*}
\]

6. Suppose that we have an infinite string, and that we know the initial position and velocity of the string for all the points \(x \in [-1, 1]\) (and we do
not have any information on \( f \) and \( g \) outside this interval). (a) For what portion of the string can we find \( u(x, t) \) at time \( t = 1 \), if the speed of wave propagation is \( c = 1/2 \)? (b) Up to what time can we say anything about any points along the string? (In other words, what is the largest value, \( t_1 \), for which we know \( u(x_1, t_1) \) for at least one point \( x_1 \)?) Hint: Find the point, \((x_1, t_1)\) whose domain of dependence for \( t = 0 \) is \([-1, 1]\).

**Solution:** We can draw a triangle with vertices at points \((-1, 0), (1, 0),\) and \((0, 2)\) in the \( x - t \) diagram. The left and right sides are characteristics with slopes \(1/c = 2\). This is the domain of dependence of point \((0, 2)\). For the point \((0, 2)\), the domain of dependence for \( t = 0 \) is the interval \([-1, 1]\). For any point with \( t > 2 \), the domain of dependence at \( t = 0 \) will not be contained in \([-1, 1]\). This means that for \( t > 2 \), we cannot find the solution \( u(x, t) \) for any value of \( x \) (we do not have enough information for it). This answers question (b). For (a), we draw a horizontal line at \( t = 1 \). It crosses the triangle at the points \((-1/2, 1)\) and \((1/2, 1)\). It is easy to check that the domains of dependence of all the points on this line with \( x \in [-1/2, 1/2] \) are contained in the large triangle. In particular, for \( t = 0 \), their domains of dependence are inside \([-1, 1]\). Therefore, for \( t = 1 \), we can write down the solution for points with \( x \in [-1/2, 1/2] \).

7. Consider a wave equation on an infinite line,
\[
\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^4} \frac{\partial^2 u}{\partial x^2} = 0.
\]
Find the characteristics though the point \((0, 3)\). Draw the domains of dependence and influence of the point \((0, 3)\) (for \( t \geq 0 \)).

**Solution:** The equations of characteristics are \( t = \pm x^3/3 + 3 \). The domain of dependence is below the two curves. The domain of influence is above.

8. Is the equation
\[
\frac{\partial^2 u}{\partial t^2} + 4 \frac{\partial^2 u}{\partial x \partial t} - \frac{9}{4} \frac{\partial^2 u}{\partial x^2} = 0
\]
hyperbolic, elliptic or parabolic (explain)? Find the general equations for characteristics if possible.