§ 11.1. Sequences.

**Definition:** Sequences: A **LIST** of numbers.

**Notation:** \( \{a_n\} \) or \( \{a_n\}_{n=1}^{\infty} \)

\[ a_1, a_2, a_3, a_4, \ldots, a_n, \ldots \]

**Remark 1:** Think \( a_n \) as Function \( f \), \( n \) plays the role of \( x \).

\[ a_n \quad \text{fix} \]

**Remark 2:** \( \text{LIST vs Collection/Union} \)

\( \text{of numbers}\) \( \text{of numbers}\)

\( \text{LIST has a definite ORDER. The order is} \)

\( \text{labeled by index } n=1, 2, \ldots \)

\[ n: 1, 2, 3, 4, 5, \ldots \]

\[ a: a_1, a_2, a_3, a_4, a_5, \ldots \]

**Examples of sequence of numbers.**

**eg. 1:** Investment at bank: $1,000 is deposited into bank at 6% annual interest rate.

After 1 year: Money in the bank

\[ 1,000 + 1,000 \times 0.06 = 1,000(1 + 0.06) = 1,000 \times 1.06 \]

\( a_1 \)

After 2 years:

\[ 1,000 \times 1.06 + 1,000 \times 1.06 \times 0.06 = 1,000 \times 1.06 \times 1.06 \]

\( a_2 \)

\[ \vdots \]

After \( n \) years:

\[ \ldots = 1,000 \times 1.06^n \]

\( a_n \)
So your investment after \( n \) years will be worth

\[
A_n = 1000 \times 1.06^n, \quad n=1, 2, \ldots.
\]

You can write \( \varnothing \) it as

\[
1000 \times 1.06, \quad 1000 \times 1.06^2, \quad 1000 \times 1.06^3, \ldots.
\]

eg. 2: Constant sequence (all \( a_n \) are the same number)

\[
\begin{align*}
\{ a_n = 0 \}_n & : 0, 0, 0, \ldots, 0, \ldots. \\
\{ a_n = 5 \}_n & : 5, 5, 5, \ldots, 5, \ldots.
\end{align*}
\]

eg. 3: Some non-trivial examples

Sequence

\[
\{ a_n = \frac{2}{n} \}_n : \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \ldots, \frac{2}{100}, \ldots
\]

\[
A_n = (-1)^n
\]

\[
-1, \quad 1, \quad -1, \quad 1, \quad -1, \quad \ldots, \quad -1, \ldots
\]

\[
A_n = (-1)^{n+1}
\]

\[
1, \quad -1, \quad 1, \quad -1, \quad 1, \quad \ldots, \quad 1, \ldots
\]

eg. 4: Relation with function \( f(x) \) and its graph

\[
\begin{align*}
f(x) &= e^x \\
f(1) &= e, \quad f(2) = e^2, \ldots, \quad f(n) = e^n, \ldots
\end{align*}
\]

\[
A_n = e^{-n}
\]

\[
\{ e^{-1} \}_n^\infty : e^{-1}, e^{-2}, e^{-3}, \ldots
\]

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.8]
\draw[thick,->] (-1,0) -- (6,0) node[below] {\( x \)};
\draw[thick,->] (0,-1) -- (0,6) node[left] {\( y \)};
\draw[very thick] (0,0) -- (6,0);
\draw[very thick] (0,0) -- (0,6);
\foreach \i in {1,2,3,4,5}
\draw[fill] (\i,0) circle (2pt);
\foreach \i in {1,2,3,4,5}
\draw[fill] (0,\i) circle (2pt);
\end{tikzpicture}
\end{array}
\]
Question 1: Find the limit: \( \lim_{x \to \infty} e^{-x} = \left( e^{-\infty} \right) = 0 \)

Question 2: Find the limit: \( \lim_{n \to \infty} e^{-n} = \left( e^{-\infty} \right) = 0 \)

Most important types of problems in exams:
Find the limit of a sequence \( \{a_n\} \), and determine whether \( \{a_n\} \) is CONVERGENT or DIVERGENT.

Definition: If \( \lim_{n \to \infty} a_n = L \) (a finite number), then we say the sequence \( \{a_n\} \) converges to \( L \).

Otherwise (\( \lim_{n \to \infty} a_n \) DNE on \( \infty \)), we say \( \{a_n\} \) diverges.

\( a_n = 2 \), \( \lim_{n \to \infty} a_n = 2 \), \( \{a_n\} \) converges to 2.

\( a_n = \frac{2}{n} \), \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2}{n} = \frac{2}{\infty} = 0 \), \( \{a_n\} \) converges to 0.

\( a_n = (-1)^n \), \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n \) does not exist, \( \{a_n\} \) is divergent.

\( a_n = 2 + \sin \left( \frac{n}{10} \right) \), \( \lim_{n \to \infty} a_n = 2 + \sin \left( \frac{3}{10} \right) = 2 + \sin 0 = 2 \) \( \{a_n\} \) converges to 2.

\( a_n = n \), \( \lim_{n \to \infty} a_n = \infty \) \( \{a_n\} \) is divergent.

Remark: If THE GRAPH of \( a_n \) has a horizontal asymptote (at \( \infty \)) then \( a_n \) converges to that asymptote. \( \lim_{n \to \infty} a_n = \infty \) as \( \frac{3n}{n} \) by L'Hopital's Rule.

Otherwise, it is divergent.
\* Theorem: If \( \lim \{ a_n \} = 0 \), then \( \lim a_n = 0 \) (Squeeze Theorem)

Properties of limit of sequence. (Limit law for sequence.)

If \( \lim a_n = A \), \( \lim b_n = B \), and \( c \) is a constant, then,

1. \( \lim (a_n + b_n) = \lim a_n + \lim b_n = A + B \)
2. \( \lim (a_n - b_n) = \lim a_n - \lim b_n = A - B \)
3. \( \lim c \cdot a_n = c \cdot \lim a_n = c \cdot A \) \( \left( \lim \frac{a_n}{c} = \frac{A}{c} \right) \), \( \lim c = c \).
4. \( \lim (a_n \cdot b_n) = \lim a_n \cdot \lim b_n = A \cdot B \).
5. \( \lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} = \frac{A}{B} \) if \( B \neq 0 \). (\*\*\*\*)
6. \( \lim a_n^p = \left( \lim a_n \right)^p \) if \( p > 0 \), \( a_n > 0 \)

Exercise: Take \( a_n = 1 + \frac{1}{n} \), \( b_n = 3 - \frac{2}{n} \). Check all above properties 1-6.

[Monotonic Sequence Theorem] (Sper hard: \*\*\*\*\*\*).

1. If \( a_n \) is increasing \( (a_n \leq a_{n+1}, a_1 < a_2 < a_3 \ldots \), the graph is increasing \) and \( a_n \) is bounded from above \( (\text{there is number } M \text{ st. } a_n \leq M) \), then \( a_n \) is convergent.

2. If \( a_n \) is decreasing \( (a_n > a_{n+1}, a_1 > a_2 > a_3 \ldots \), the graph is decreasing \) and \( a_n \) is bounded from below \( (\text{there is number } m \text{ st. } a_n \geq m) \), then \( a_n \) is convergent.