§ 11.4. Comparison Tests.

Motivation: Some of the series look quite similar to (but not quite) p-series or geometric series, e.g., \( \sum_{n=1}^{\infty} \frac{1}{n^{3}+8} \), \( \sum_{n=1}^{\infty} \frac{6^n}{5^n-1} \).

We want to compare them with a standard p-series and geometric series, such as \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), \( \sum_{n=1}^{\infty} \frac{6^n}{5^n} \). Then determine the convergence/divergence of these series based on our knowledge of the previous one.

The comparison Test. Thus (9), 10, 13, 29, 16.

Suppose \( a_n, b_n \) are positive and \( a_n \leq b_n \) for all \( n \).

Then \( \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n \). Further more,

(i) If \( \sum_{n=1}^{\infty} b_n \) is convergent, then \( \sum_{n=1}^{\infty} a_n \) is convergent.

(The bigger one is convergent, then the smaller one is also convergent)

(ii) If \( \sum_{n=1}^{\infty} a_n \) is divergent, then \( \sum_{n=1}^{\infty} b_n \) is divergent.

(The smaller one is divergent, then the bigger one is also divergent)

e.g. 1) \( \frac{1}{n^3+8} \leq \frac{1}{n^3} \) for all \( n \), then \( \sum_{n=1}^{\infty} \frac{1}{n^3+8} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \) (bigger one)

Because \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) is a p-series with \( p=3 > 1 \), \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) is convergent. Then \( \sum_{n=1}^{\infty} \frac{1}{n^3+8} \) Conv.

2) \( \frac{6^n}{5^n-1} \geq \frac{6^n}{5^n} \) for all \( n \Rightarrow \sum_{n=1}^{\infty} \frac{6^n}{5^n-1} \geq \sum_{n=1}^{\infty} \frac{6^n}{5^n} \) (smaller one)

Because \( \sum_{n=1}^{\infty} \frac{6^n}{5^n} \) is a geometric series with \( r=\frac{6}{5} > 1 \), \( \sum_{n=1}^{\infty} \frac{6^n}{5^n} \) is divergent. Then \( \sum_{n=1}^{\infty} \frac{6^n}{5^n-1} \) divergent.

(The smaller one)
Linear Rule: e.g. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n=1}^{\infty} \frac{5}{n} = 5 \cdot \sum_{n=1}^{\infty} \frac{1}{n}$ is also divergent.

In general, if $\sum_{n=1}^{\infty} bn$ is conV/DIV, then for any number $c \neq 0$, $an = c \cdot bn$

$$\sum_{n=1}^{\infty} an = \sum_{n=1}^{\infty} cbn = c \cdot \sum_{n=1}^{\infty} bn$$

is conV/DIV.

The condition $an = c \cdot bn \iff \frac{an}{bn} = c$ can be generalized to the following

The Limit Comparison Test (L.C.T.)

Suppose $an$, $bn$ are positive and

$$\lim_{n \to \infty} \frac{an}{bn} = C \neq 0.$$ 

If $\sum_{n=1}^{\infty} bn$ is convergent, then $\sum_{n=1}^{\infty} an$ is also convergent. (divergent)

Example: $\sum_{n=1}^{\infty} \frac{n+1}{n^2 + n}$ convergent or not?

Solution: $an = \frac{n+1}{n^2 + n}$

Step 1: Find correct $bn$ to compare with $an$.

(Hint: Focus on the "dominant" term of $an$), $bn = \frac{n}{n^2} = \frac{1}{n}$

Step 2: take limit of $\lim_{n \to \infty} \frac{an}{bn} = \lim_{n \to \infty} \frac{(n+1) \cdot n^2}{n^2 + n} = \lim_{n \to \infty} \frac{n^2}{n^2} \cdot \frac{n+1}{n+1} = \lim_{n \to \infty} \frac{n^2}{n^2} = 1 \neq 0$

Step 3: Apply L.C.T.

Since $\sum_{n=1}^{\infty} bn = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, (p-series $p=2$) $\sum_{n=1}^{\infty} an$ is also convergent.

According to L.C.T. $\sum_{n=1}^{\infty} \frac{n+1}{n^2 + n}$ is also convergent.
Remark: For most problems, an answer purely consists of polynomials (power functions) of $n$ or purely consists of exponential functions of $n$. There is a quick way to find $b_n$ and the limit of $\frac{a_n}{b_n}$ is forever 1.

Rule: Drop the lower order terms of $a_n$. The remainder will be $b_n$.

Purely Polynomials of $n$: Ex 3, 4, 5, 6, 11, 12, 14, 17, 18, 19, 20, 21, 22, 23, 26 to 114

**Example:**
$$
\sum_{n=1}^{\infty} \frac{2n-1}{n^3 + 3n^2 + 2}, \quad a_n = \frac{2n-1}{n^3 + 3n^2 + 2}, \quad \text{lower order terms: } -1.
$$

$$
b_n = \frac{2n}{n^3}
$$

$$
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2n-1}{2n^4 + 6n^3 + 3n^2 + 2} = 1.
$$

Since $\sum_{n=1}^{\infty} \frac{2n}{n^3}$ is convergent, $\sum_{n=1}^{\infty} \frac{2n-1}{n^3 + 3n^2 + 2}$ is also convergent.

**Example:**
$$
\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}, \quad \text{lower order terms: } \frac{1}{n^2 + 1}.
$$

$$
b_n = \frac{\sqrt{n}}{n^3} = \frac{\sqrt{n}}{\sqrt{2n^3}} = \frac{n^{\frac{1}{2}}}{\sqrt{2} \cdot n^\frac{3}{2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{n^2} \quad \frac{3}{2} - \frac{1}{2} = 1
$$

$$
\lim_{n \to \infty} \frac{a_n}{b_n} = 1.
$$

Because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is divergent, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$ is also divergent.
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**n**!**n**! + e.g. \( \sum_{n=1}^{\infty} \frac{4^n}{3^{n-2}} \). \( a_n = \frac{4^n}{3^{n-2}} \) lower order term: -2.

Pick \( b_n = \frac{4^n}{3^n} \).

Then \( \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{4^n}{3^{n-2}} \cdot \frac{3^n}{4^n} = \lim_{n \to \infty} \frac{3^n}{3^{n-2}} = 1 \).

Because \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4^n}{3^n} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{3}\right)^n \) C.S. \( r = \frac{4}{3} > 1 \).

is divergent, therefore \( \sum_{n=1}^{\infty} \frac{4^n}{3^{n-2}} \) is also divergent.

Some difficult problems via expansion Tests.

\( \therefore \sum_{n=1}^{\infty} \frac{1}{n!} \), \( a_n = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n} < \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 2} = \frac{1}{2^{n+1}} \)

Therefore, \( \sum_{n=1}^{\infty} \frac{1}{n!} \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \). C.S. with \( r = \frac{1}{2} \) convergent.

\( \therefore \sum_{n=1}^{\infty} \frac{1}{n!} \) is convergent.

\( \sum_{n=1}^{\infty} \frac{e^n}{n!} \), \( a_n = \frac{e^n}{n!} \), let \( b_n = \frac{1}{n} \). Then

\( \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{e^n}{n} = \lim_{n \to \infty} e^n = e^0 = 1 \).

Because \( \sum_{n=1}^{\infty} \frac{1}{n!} \) is divergent,

therefore, \( \sum_{n=1}^{\infty} \frac{e^n}{n!} \) is also divergent.