3.2. Volumes.

Three types: (Three key formulas)

1) Disk (cross-section): \[ V = \pi \int_a^b \text{Radius} \, dx \]

Solid of revolution:

2) Washer: \[ V = \pi \int_a^b [\text{Outer Radius}^2 - \text{Inner Radius}^2] \, dx \]

3) Pyramid (square cross-section): \[ V = \frac{1}{3} \int_a^b \text{Length}^2 \, dx \] (side)

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**Diagram:**

- Point → Length
- Area → Volume

**Differentiation → Integration**

- Area: \[ \int_a^b \text{Area} \, dx = \int_a^b \text{Length} \, dx \]

**Generalization (Area → Volume):**

- Volume: \[ \int_a^b \text{Volume} \, dx = \int_a^b \text{Area of the cross-section} \, dx \]
we only deal with two types of cross-sections.

- disk
  (Washer: two disks)

- square

region bounded by
\[ y = \sqrt{x} \] from \( x=0 \) to \( x=1 \)

rotating with respect to \( x \)-axis.

at position \( x \), the cross-section is a DISK with
radius \( r = \sqrt{x} \).

Area = \( \pi \cdot r^2 = \pi \cdot (\sqrt{x})^2 \).

Volume = \( \int_0^1 \pi \cdot (\sqrt{x})^2 \, dx \) (Set up integral only)
The solid (outer) generated by $y = -\sqrt{x}$ is I.
The (inner) solid generated by $y = \sqrt{x}$ is II.
The shadow solid = I - II.
Cross-section is a comb of two disks.
Area = $\pi \cdot \text{Outer} - \pi \cdot \text{Inner}$
Volume $\pi \int_0^1 [(\sqrt{x})^2 - x^2] \, dx$.
(Setup only)

Pyramid: The same region as above (as base). The cross-section is a square perpendicular to $x$-axis with side in $x$-$y$ plane
$L(x) = \sqrt{x} - x$.
Area of square = $L(x) = (\sqrt{x} - x)^2$
Volume $\int_0^1 (\sqrt{x} - x)^2 \, dx$. 
Rotating about \( y \)-axis:

\[ y = \sqrt{x}, \quad y = 0 \text{ to } y = 1 \]

rotated about \( y \)-axis

\[
\text{Volume} = \pi \int_0^1 y^2 dy
\]

Rotating about \( x \)-axis parallel \( x \)-axis (along \( x \)-direction)

\[ y = 1, \quad (w \text{ y } = -1) \]

Region: \( y = \sqrt{x}, \quad y = x, \quad x = 1 \)

Cross-section:
- Outer Radius = 1 - \( x \)
- Inner Radius = 1 - \( \sqrt{x} \)

\[
\text{Volume} = \pi \int_0^1 [(1-x)^2 - (1-\sqrt{x})^2] dx
\]