October 16, 2015

Midterm 1-Form A

Name: ____________________________  Student ID: __________________

Instructions.

1. This exam consists of 7 problems and 100 total points. You have 60 minutes to take the exam.

2. Read the directions for each problem carefully and answer all parts of each problem.

3. Show all the steps of your work clearly and in order, and circle your final answers.

4. Justify your answers algebraically whenever possible.

5. No notes, phones and calculators.

Answer Key
Problem 1) [6 points]

(a) (2 points) Graph the function \( f(x) = x^2 \), \(-1 \leq x \leq 3\). Estimate the area under the graph of \( f \) using four approximating rectangles and taking the sample points to be left endpoints.

\[
\begin{align*}
\Delta x &= \frac{3-(-1)}{4} = 1 \\
x &= -1, \quad x = 0, \quad x = 1, \quad x = 2 \\
\text{height: } f(-1) &= 1, \quad f(0) = 0, \quad f(1) = 1, \quad f(2) = 4 \\
\text{Estimate} &= \Delta x \left[ f(-1) + f(0) + f(1) + f(2) \right] \\
&= 1 \cdot (1 - 1 + 1 + 0 + 1 + 1 + 1.4) \\
&= \boxed{6}
\end{align*}
\]

(b) (2 points) Write \( \int_{-1}^{3} x^2 \, dx \) as the limit of Riemann sums (right endpoints rule).

\[
\begin{align*}
n, \quad \Delta x &= \frac{3-(-1)}{n} = \frac{4}{n}, \quad a = -1, \quad b = 3 \\
x_i &= a + i \cdot \Delta x = -1 + i \cdot \frac{4}{n}, \quad i = 1, 2, \ldots, n \\
f(x_i) &= x_i^2 = (-1 + i \cdot \frac{4}{n})^2 \\
\text{Riemann Sum} &= \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \sum_{i=1}^{n} (-1 + i \cdot \frac{4}{n}) \cdot \frac{4}{n} \\
\int_{-1}^{3} x^2 \, dx &= \lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} (-1 + i \cdot \frac{4}{n}) \cdot \frac{4}{n}
\end{align*}
\]

(c) (2 points) Evaluate the integral in (b) by using Fundamental Theorem of Calculus.

\[
\begin{align*}
\int_{-1}^{3} x^2 \, dx &= \left. \frac{1}{2+1} \cdot x^{2+1} \right|_{-1}^{3} \\
&= \left. \frac{1}{3} \cdot x^3 \right|_{-1}^{3} \\
&= \left. \frac{1}{3} \cdot 3^3 \right|_{-1}^{3} - \left. \frac{1}{3} \cdot (-1)^3 \right|_{-1}^{3} \\
&= \boxed{\frac{28}{3}}
\end{align*}
\]
Problem 2) [10 points]

(a) (6 points) Below is the graph of function \( f(t) \). Let \( g(x) = \int_0^x f(t) \, dt \).

Find \( g(1), g'(3), g''(4) \).

\[
g(1) = \int_0^1 f(t) \, dt = \text{area of } \triangle \quad I = \frac{1}{2} \times 1 \times 3 = \frac{3}{2}
\]

\[
g(3) = f(3) = -1
\]

\[
g''(4) = f''(4) = 0 \quad \text{(slope 0 at 4)}
\]

(b) (4 points) Let

\[
h(u) = \int_{2 \tan^{-1} u}^0 \cos(e^x) \, dx.
\]

Find \( h'(u) = \frac{dh}{du} \).

\[
h(u) = \int_{2 \tan^{-1} u}^0 \cos(e^x) \, dx = -\int_0^{2 \tan^{-1} u} \cos(e^x) \, dx
\]

\[
h'(u) = -\cos(e^{2 \tan^{-1} u}) \cdot (2 \tan^{-1} u)'
\]

\[
= -\cos(e^{2 \tan^{-1} u}) \cdot 2 \cdot \frac{1}{1 + u^2}
\]
Problem 3]... [8 points]

(a)(3 points) If \( \int_0^3 f(x) \, dx = 2 \), \( \int_0^3 g(x) \, dx = 5 \), find \( \int_0^3 [f(x) - 2g(x)] \, dx \).

\[
\int_0^3 g(x) \, dx = 5 \implies \int_0^3 g(x) \, dx = -\int_0^3 g(x) \, dx = -5.
\]

\[
\int_0^3 [f(x) - 2g(x)] \, dx = \int_0^3 f(x) \, dx - 2 \int_0^3 g(x) \, dx
\]

\[
= 2 - 2 \cdot (-5) = 12.
\]

(b)(3 points) If \( F(10) = 3 \), \( F(2) = 0 \), find \( \int_2^{10} 3F'(x) \, dx \).

\[
\int_2^{10} 3F'(x) \, dx = 3 \int_2^{10} F'(x) \, dx
\]

\[
= 3 \cdot F(x) \bigg|_{2}^{10}
\]

\[
= 3 \cdot F(10) - 3 \cdot F(2) = 3 \cdot 3 - 0 = 9.
\]

(c)(2 points) If \( \int_0^4 f(x) \, dx = 3 \), find \( \int_0^4 2xf(x^2) \, dx \).

\[
u = x^2, \quad du = 2x \, dx.
\]

\[
\int_0^4 2x \cdot f(x^2) \, dx
\]

\[
= \int_0^4 f(x^2) \cdot 2x \, dx
\]

\[
= \int_0^4 f(u) \cdot du
\]

\[
= 3.
\]
Problem 4]... [20 points] Evaluate the following definite integrals.

(a) (10 points)
\[
\int_0^1 e^{u^2} du = \int_0^1 e^u \cdot e^2 \cdot du = e^2 \int_0^1 e^u \cdot du = e^2 \cdot e^1 - e^2 \cdot e^0 = [e^3 - e^2]
\]

(b) (10 points)
\[
\int_1^e \frac{2\sqrt{\ln x}}{x} dx = \int_{\ln 1}^{\ln e} \frac{x}{x} \cdot \frac{1}{\sqrt{u}} \cdot du = \int_{\ln 1}^{\ln e} \frac{1}{\sqrt{u}} \cdot du = \left[ 2\sqrt{u} \right]_{u=0}^{u=1} = 2\sqrt{1} - 2\sqrt{0} = \left[ \frac{2}{3} \right]
\]
Problem 5] [20 points] Evaluate the following indefinite integrals.

(a) (10 points)
\[ \int \frac{(\sqrt{x} + 1)^2}{x} \, dx \]
\[ \int \frac{(\sqrt{x} + 1)^2}{x} \, dx = \int \frac{x + 2\sqrt{x} + 1}{x} \, dx \]
\[ = \int 1 + 2 \cdot x^{-\frac{1}{2}} + \frac{1}{x} \, dx \]
\[ = x + 2 \cdot \frac{1}{-\frac{1}{2} + 1} \cdot x^{-\frac{1}{2} + 1} + \ln|x| + C \]
\[ = \left( x + 4 \cdot x^{\frac{1}{2}} + \ln|x| \right) + C \]

(b) (10 points)
\[ \int \frac{x^3}{x^4 + 2} \, dx \]
\[ u = x^4 + 2 \]
\[ du = 4 \cdot x^3 \, dx \]
\[ \frac{du}{4} = x^3 \, dx \]
\[ \int \frac{x^3 \, dx}{x^4 + 2} \]
\[ = \int \frac{1}{u} \cdot \frac{1}{4} \, du \]
\[ = \frac{1}{4} \int \frac{1}{u} \, du \]
\[ = \frac{1}{4} \ln|u| + C \]
\[ = \left( \frac{1}{4} \ln|x^4 + 2| \right) + C \]
Problem 6]...[16 points]

Let $R$ be the region bounded by $y = \sqrt{x-2}$ and $y = x - 2$.

(a) (10 points) Sketch $R$ and find its area.

Intersection points

\[
\begin{align*}
\begin{cases}
    y = \sqrt{x-2} \\
    y = x - 2
\end{cases}
\Rightarrow \quad & x - 2 = \sqrt{x-2} \\
\Rightarrow \quad & x^2 - 4x + 4 = x - 2 \\
\Rightarrow \quad & x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0 \Rightarrow x=2, x=3
\end{align*}
\]

Area = \[ \int_{2}^{3} (x-2) - (\sqrt{x-2}) \, dx \]

\[
= \left[ \frac{1}{3} u^3 - u^2 \right]_{2}^{3} = \frac{2}{3} \cdot 3^2 - \frac{2}{3} \cdot 2^2 = 2 \cdot \frac{2}{3} - \frac{2}{3} \cdot 1 = \frac{4}{3} - \frac{2}{3} - 0 = \frac{2}{3}
\]

(b) (6 points) Let the solid $S$ be generated by revolving the region $R$ (the region bounded by $y = \sqrt{x-2}$ and $y = x - 2$) about $y = -2$. Set up an integral to express the volume the solid $S$.

You DO NOT need to evaluate the integral.

\[
\text{Radius} = \sqrt{x-2} - (-2) = \sqrt{x-2} + 2
\]

\[
\text{Inner} = x - 2 - (-2) = x
\]

Volume = \[ \pi \int_{2}^{3} \text{Radius}^2 - \text{Inner}^2 \, dx \]

\[
= \pi \left[ \int_{2}^{3} (\sqrt{x-2} + 2)^2 - x^2 \right] dx
\]
Problem 7... [20 points] A particle moves along a line so that its velocity (in meters per second) at time $t$ seconds is given by $v(t) = 2 \cos t$.

(a) (5 points) Find the displacement of the particle during the time period $0 \leq t \leq \pi$

\[
\text{Displacement} = \int_{0}^{\pi} v(t) \, dt = \int_{0}^{\pi} 2 \cos t \, dt = 2 \sin t \bigg|_{0}^{\pi} = 2 \sin \pi - 2 \sin 0 = 0 \quad \text{(meters)}
\]

(b) (5 points) Set up the integral for the average velocity (the average value of function $v(t)$) during the time period $0 \leq t \leq \pi$ and find its value.

\[
\text{Ave Velocity} = \frac{1}{\pi} \cdot \int_{0}^{\pi} v(t) \, dt = \frac{1}{\pi} \cdot 0 = 0 \quad \text{(meters/second)}
\]
(c) (10 points) Find the total distance traveled by the particle during the time period $0 \leq t \leq \pi$.

\[
\text{Distance} = \int_0^\pi |v(t)| \, dt
\]

\[
= \int_0^\pi |2\cos t| \, dt
\]

\[
= \int_0^\frac{\pi}{2} 2\cos t \, dt + \int_{\frac{\pi}{2}}^\pi (-2\cos t) \, dt
\]

\[
= 2\sin t \bigg|_0^{\frac{\pi}{2}} - 2\sin t \bigg|_{\frac{\pi}{2}}^\pi
\]

\[
= 2\sin \frac{\pi}{2} - 0 - \left[2\sin \pi - 2\sin \frac{\pi}{2}\right]
\]

\[
= 2 - 0 - [0 - 2]
\]

\[
= 4 \quad \text{meters}
\]