October 16, 2015

Midterm 1-Form B

Name: ___________________________ Student ID: ___________________________

Instructions.

1. This exam consists of 7 problems and 100 total points. You have 60 minutes to take the exam.

2. Read the directions for each problem carefully and answer all parts of each problem.

3. Show all the steps of your work clearly and in order, and circle your final answers.

4. Justify your answers algebraically whenever possible.

5. No notes, phones and calculators.
Problem 1] [10 points]

(a) (6 points) Below is the graph of function $f(t)$. Let $g(x) = \int_0^x f(t) \, dt$.

\[ g(1) = \int_0^1 f(t) \, dt = \text{Area } I = \frac{1}{2} \times 1 \times 2 = 1 \]

\[ g'(3) = f(3) = -1 \]

\[ g''(4) = f'(4) = 0 \quad (\text{slope } 0 \text{ at } t=4) \]

(b) (4 points) Let

\[ h(u) = \int_{2 \tan^{-1} u}^0 \sin(e^x) \, dx. \]

Find $h'(u) = \frac{dh}{du}$.

\[ h(u) = \int_{2 \tan^{-1} u}^0 \sin(e^x) \, dx = - \int_0^{2 \tan^{-1} u} \sin(e^x) \, dx \]

\[ h'(u) = - \sin(e^{2 \tan^{-1} u}) \cdot (2 \tan^{-1} u)' \]

\[ = - \sin(e^{2 \tan^{-1} u}) \cdot 2 \cdot \frac{1}{1 + u^2} \]
Problem 2]...[8 points]

(a) (3 points) If \( \int_0^3 f(x) \, dx = 1, \int_0^3 g(x) \, dx = 4 \), find \( \int_0^3 [f(x) - 3g(x)] \, dx \).

\[
\int_0^3 g(x) \, dx = -\int_0^3 g(x) \, dx = -4
\]

\[
\int_0^3 [f(x) - 3g(x)] \, dx = \int_0^3 f(x) \, dx - 3 \int_0^3 g(x) \, dx = 1 - 3 \cdot (-4) = \boxed{13}
\]

(b) (3 points) If \( F'(10) = 4, F'(2) = 0 \), find \( \int_2^{10} 4F''(x) \, dx \).

\[
\int_2^{10} 4F''(x) \, dx = 4 \int_2^{10} F''(x) \, dx = 4 \left. F(x) \right|_2^{10} = 4 \cdot F(10) - 4 \cdot F(2) = 4 \cdot 4 - 0 = \boxed{16}
\]

(c) (2 points) If \( \int_0^3 f(x) \, dx = 5 \), find \( \int_0^3 2x f(x^2) \, dx \).

\[
u = x^2, \quad du = 2x \, dx \quad \int_{x=0}^{x=3} u^{\frac{3}{2}} \int_{u=0}^{u=9} \frac{u^{\frac{3}{2}}}{x} \, dx
\]

\[
\int_0^3 2x f(x^2) \, dx = \int_0^9 f(u) \cdot du
\]

\[
= \boxed{5}
\]
Problem 3]...[6 points]

a) (2 points) Graph the function \( f(x) = x^2, \ -1 \leq x \leq 3 \). Estimate the area under the graph of \( f \) using four approximating rectangles and taking the sample points to be left endpoints.

\[
\text{Estimate} = \Delta x \left[ f(1) + f(0) + f(1) + f(2) \right]
\]

\[
= 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 4
= 6
\]

b) (2 points) Write \( \int_{-1}^{3} x^2 \, dx \) as the limit of Riemann sums (right endpoints rule.)

\[
n, \quad \Delta x = \frac{3 - (-1)}{n} = \frac{4}{n}, \quad x_i = a + i \cdot \Delta x = \frac{4i}{n}, \quad i = 1, 2, \ldots, n
\]

\[
\int_{-1}^{3} x^2 \, dx = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} \left( \frac{i}{n} \right)^2 \cdot \frac{4}{n} \right]
\]

\[
= \frac{28}{3}
\]

c) (2 points) Evaluate the integral in b) by using Fundamental Theorem of Calculus.

\[
\int_{-1}^{3} x^2 \, dx = \left. \frac{1}{3} x^3 \right|_{-1}^{3}
\]

\[
= \frac{1}{3} \cdot 3^3 - \frac{1}{3} \cdot (-1)^3
\]

\[
= \frac{28}{3}
\]
Problem 4]...[20 points] Evaluate the following indefinite integrals.

(a) (10 points)

\[
\int \frac{(\sqrt{x} + 1)^2}{x} \, dx
\]

\[
\int \frac{(\sqrt{x} + 1)^2}{x} \, dx = \int \frac{x + 2\sqrt{x} + 1}{x} \, dx
\]

\[
= \int 1 + 2\sqrt{x} + \frac{1}{x} \, dx
\]

\[
= x + 2\cdot \frac{1}{\frac{1}{2} + 1} \cdot x^{\frac{1}{2} + 1} + \ln|\sqrt{x}| + C
\]

\[
= x + 4 \cdot x^{\frac{1}{2}} + \ln|\sqrt{x}| + C
\]

(b) (10 points)

\[
\int \frac{x^3}{x^4 + 3} \, dx
\]

\[
u = x^4 + 3 , \quad du = 4x^3 \, dx
\]

\[
\frac{1}{4} \, du = x^3 \, dx
\]

\[
\int \frac{x^3 \, dx}{x^4 + 3} = \int \frac{1}{x^4 + 3} \cdot x^3 \, dx
\]

\[
= \int \frac{1}{u} \cdot \frac{1}{4} \, du
\]

\[
= \frac{1}{4} \ln|u| + C
\]

\[
= \frac{1}{4} \ln|x^4 + 3| + C
\]
Problem 5)... [20 points] Evaluate the following definite integrals.

(a) (10 points)
\[
\int_0^1 e^{u^3} \, du
= \int_0^1 e^u \cdot e^3 \, du
= e^3 \int_0^1 e^u \, du
= e^3 \cdot e^u \bigg|_0^1
= e^3 \cdot e^1 - e^3 \cdot e^0
= e^4 - e^3
\]

(b) (10 points)
\[
\int_1^e \frac{3\sqrt{\ln x}}{x} \, dx
u = \ln x, \quad \int_{x=1}^{x=e} \frac{3\sqrt{u}}{u} \, du = \int_{u=0}^{u=\ln e} \frac{3}{u^{3/2}} \, du
= 3 \cdot \frac{2}{3} \cdot u^{3/2} \bigg|_0^1
= 2 \cdot 1^{3/2} - 2 \cdot 0 = 2
\]
Problem 6]...[20 points] A particle moves along a line so that its velocity (in meters per second) at time \( t \) seconds is given by \( v(t) = 3 \cos t \).

(a) (5 points) Find the displacement of the particle during the time period \( 0 \leq t \leq \pi \)

\[
\text{Displacement} = \int_{0}^{\pi} v(t) \, dt \\
= \int_{0}^{\pi} 3 \cos t \, dt \\
= 3 \sin t \bigg|_{0}^{\pi} \\
= 3 \sin \pi - 3 \sin 0 \\
= \boxed{0} \text{ (meters)}
\]

(b) (5 points) Set up the integral for the average velocity (the average value of function \( v(t) \)) during the time period \( 0 \leq t \leq \pi \) and find its value.

\[
\text{Average Velocity} = \frac{1}{\pi - 0} \int_{0}^{\pi} v(t) \, dt \\
= \frac{1}{\pi} \cdot 0 \\
= \boxed{0} \text{ (meters/second)}
\]
(c) (10 points) Find the total distance traveled by the particle during the time period $0 \leq t \leq \pi$.

Distance $= \int_0^\pi |3\cos t| \, dt$

$= \int_0^{\frac{\pi}{2}} 3\cos t \, dt + \int_{\frac{\pi}{2}}^\pi 3\cos t \, dt$

$= 3\sin t \bigg|_0^{\frac{\pi}{2}} - 3\sin t \bigg|_{\frac{\pi}{2}}^\pi$

$= 3\sin \frac{\pi}{2} - 3\sin 0 - \left[ 3\sin \pi - 3\sin \frac{\pi}{2} \right]$

$= 3 \cdot 1 - 0 - [0 - 3]

= $6$ (meters)
Problem 7) [16 points]

Let $R$ be the region bounded by $y = \sqrt{x-3}$ and $y = x - 3$.

(a) [10 points] Sketch $R$ and find its area.

Intersection point $A$

$y = \sqrt{x-3} = x - 3$

$\Rightarrow$ $x - 3 = x^2 - 6x + 9$

$\Rightarrow$ $x^2 - 7x + 12 = 0$

$\Rightarrow$ $(x-3)(x-4) = 0$

$\Rightarrow$ $x = 3$ or $x = 4$.

Area $= \int_{3}^{4} \sqrt{x-3} - (x-3) \, dx$

$= \int_{3}^{4} u - u \cdot du = \frac{2}{3} u^\frac{3}{2} - \frac{1}{2} u^2 |_1^4$

$= \frac{3}{3} - \frac{1}{2} - 0 = \boxed{\frac{1}{6}}$

(b) [6 points] Let the solid $S$ be generated by revolving the region $R$ (the region bounded by $y = \sqrt{x-3}$ and $y = x - 3$) about $y = -3$. Set up an integral to express the volume the solid $S$.

You DO NOT need to evaluate the integral.

$\text{Rearr} = \sqrt{x-3} - (-3) = \sqrt{x-3} + 3$

$\text{Rinner} = x - 3 - (-3) = x$

$\text{Volume} = \int_{3}^{4} \pi \left( \sqrt{x-3} + 3 \right)^2 - x^2 \, dx$

$\boxed{\pi \int_{3}^{4} \left( \sqrt{x-3} + 3 \right)^2 - x^2 \, dx}$